LATTICE THEORY
The Atomic Age in Mathematics

Books with
text by LILLIAN R. LIEBER
drawings by HUGH G. LIEBER

Non-Euclidean Geometry
Galois and the Theory of Groups
The Einstein Theory of Relativity
The Education of T. C. Mits
Mits, Wits and Logic
Mathematics for the Two Billion
Infinity

Books of drawings by Hugh G. Lieber
Goodbye Mr. Man, Hello Mr. NEWman
(with Introduction by Lillian R. Lieber)
Comédie Internationale

A Bouquet of Qwordoodles for Elarel
(Verse and drawings by Hugh G. Lieber)

LATTICE THEORY
The Atomic Age in Mathematics

Text by
LILLIAN R. LIEBER

Drawings by
HUGH G. LIEBER

GALOIS INSTITUTE OF MATHEMATICS AND ART
BROOKLYN, N. Y.

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DEDICATION

To Albert Einstein,
the great mind and heart of our time,
who, by his discovery of
ATOMIC ENERGY,
so beautifully expressed in
his amazing little equation
$E = MC^2$,
has given to Man
a NEW DIRECTION —
a NEW VISION of ABUNDANCE,
a NEW vision of HEALTH —
and has made
WAR itself IMPOSSIBLE.

May the Lord bless thee and keep thee
May the Lord let His face shine unto thee
and show thee favor,
May the Lord lift up
His countenance unto thee
and give thee peace.

Galois 6-11-65 5.95

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ACKNOWLEDGMENTS

I am deeply grateful to Professor Garrett Birkhoff, a great creative mathematician and Head of the Department of Mathematics at Harvard University, for encouraging me to write this little book, for reading it in manuscript and making some very valuable suggestions, for saying it is "a charming addition to the literature on lattice theory --- to which it should win many converts and friends", and most especially for being as enthusiastic about H.G.'s drawings as I am.

My thanks are due also to Professor Ernest Nagel of Columbia University, a well-known philosopher and logician, for reading Part III, "Behind the Scenes", for making several valuable suggestions, and for agreeing with the point of view expressed there, namely, that Science, regarded as an institution (or form of human activity), is NOT AMORAL, that this idea must be made EXPLICIT, and that this must be repeated over and over again, until everyone hears it and can examine it and think about it, and NOT continue to repeat the clichés: "Science is amoral", and "Scientists are a menace bent on destroying mankind"!

L.R.L.

-6-5
0 1 2 3 4 5 6 7 8 9 10 11
P1
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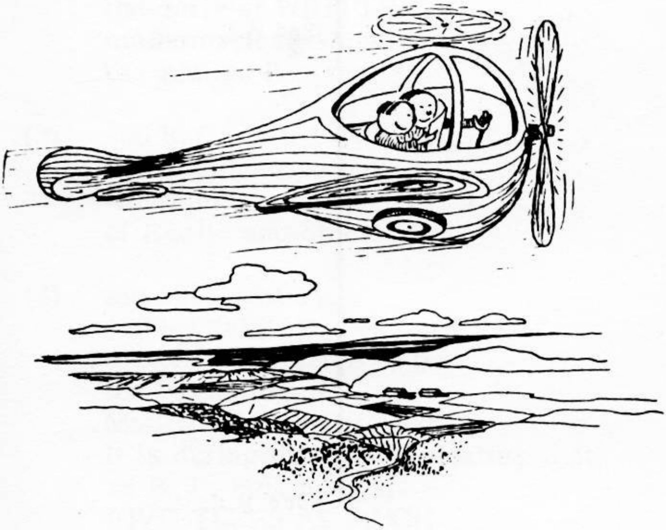
Part I

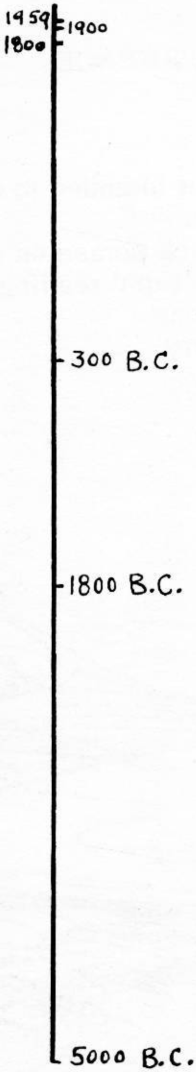
The Nineteenth Century:

The NEW FREEDOM

PREFACE

This is not intended to be
free verse.
Writing each phrase on a separate line
facilitates rapid reading,
and everyone
is in a hurry
nowadays.




1959
1900
1800
300 B.C.
1800 B.C.
5000 B.C.

I. Introduction.

In order to appreciate properly
the power and beauty of
Lattice Theory,
it is well to take a brief look at
the history of mathematics as depicted in
the single line on the opposite page!
This shows a few most significant dates: *

(1) 5000 B.C. which, so far as we know,
is the approximate date of
the earliest known mathematics; *

(2) 1800 B.C. is the date of
the earliest WRITTEN
mathematical document,
the Ahmes Papyrus; *

(3) 300 B.C. is the well-known date
in the golden period of
Greek mathematics,
of Euclid and others; *

(4) the 19th century,
1800-1900 A.D.,
may be considered to be
the beginning of
Modern Mathematics. *
It is during this single century that,
as E.T. Bell * says,
FIVE TIMES AS MUCH
CREATIVE WORK in mathematics was done
as in all of previous history!

---

* See the wonderful books by E. T. Bell:
"The Development of Mathematics" (McGraw Hill Book Co.)
"Men of Mathematics", p. 17 (Simon and Schuster)

It is interesting to consider
why this was so!

The fact is that
it was at the beginning of
the 19th century that
THE great discovery of
MODERN mathematics
was made,
namely,
the realization that
the postulates of a mathematical system
are NOT "self-evident truths"
(as they were considered to be before that)
but are merely
MAN-MADE ASSUMPTIONS.
And it was this
CHANGE of ATTITUDE toward
the NATURE of postulates
that made it possible to
MAKE new postulate sets,
since these are only MAN-MADE
and therefore subject to change by man.
HOW this change of attitude came about
can be appreciated by
a consideration of
the very interesting history of
Euclid's famous Parallel Postulate! *

And so
the 19th century saw the development of
a NEW FREEDOM
leading to many new postulate sets,
resulting in many new
branches of mathematics,

---

* See (1) A. Church: "The Euclidean Parallel Postulate"
(2) H.G. and L.R. Lieber: "Human Values of Modern Mathematics"
both published by the Galois Institute of Mathematics and Art.

geometries, algebras,
and even logics (in the plural!),
which, by means of
GENERALIZATION and ABSTRACTION,
turned out to be
new, powerful tools of thought.

Some of these will be
briefly described in subsequent chapters.

And now,
in the years from 1900 to 1959,
especially since 1930,
we see what might be called
the ATOMIC AGE of mathematics —
an age in which
ORDER and STRUCTURE
have become objects of study —
just as, in Physics,
the STRUCTURE of the ATOM
occupies the center of attention.

This study of
ORDER and STRUCTURE
in mathematics
has led to the amazing
LATTICE THEORY,
developed by
the young, brilliant mathematician,
Garrett Birkhoff of Harvard University,
and others. *
The abundance of lattices in mathematics
was apparently not realized
before Dedekind (1897).
Following Dedekind,
Emmy Noether,
the great woman mathematician,
stressed their importance in algebra.

---

* See G. Birkhoff: "Lattice Theory" (1948)
p. 17.

Their importance in other domains
seems to have been discovered
INDEPENDENTLY by
Fritz Klein (1929),
K. Menger (1928), and
G. Birkhoff. *

In subsequent chapters we shall see
what is a lattice,
what are the different kinds of lattices,
how they are applied to
various algebras, geometries, etc.,
how they help us to see
connections between various branches of mathematics,
thus exhibiting their
COMMON STRUCTURE,
in spite of the fact that
they SEEM so DIFFERENT.
To emphasize this last point,
especially for readers who
do not like the "abstract" and
think it is "impractical",
let me quote from an article
"Operations Research" by
Philip M. Morse,
in the first issue of the
IRE Transactions on Military Electronics,
March, 1957, pages 7 and 8:

In speaking of an important chapter of
modern mathematics,
known as Linear Programming,
he says:
- - - - "the optimization of
crude-oil-cracking production
is a linear programming problem.
An oil company can produce
various proportions of

---

* See G. Birkhoff: "Lattice Theory" (1948).

fuel oil,
lubricating oil, and
gasoline
from their cracking plants,
depending on
the crude used, and on
the cracking process used.
But crudes differ in price,
and cracking processes differ in cost.
If the company has
orders for definite quantities of end products,
what amounts
of which crudes
shall it buy
and which processes shall it use
in which cracking plants
to produce the required products
at least cost,
subject to limitations of
supply of crudes
and outputs of their plants?
This is a straightforward
linear programming problem,
and some oil company engineers
have been doing a good job solving it.
These engineers did not call what they were doing
operations research,
some of them had not heard of the term.
But neither were these engineers aware that
many other problems in
their own company's operations
were also amenable to this same analysis.
Their training and their outlook
had been too specialized to see
the wider possibilities.
The value of the concept of
operations research
to this oil company
lies in making the company research men aware
that the mathematics they have been using

can be applied to many more
operational problems
than they had hitherto conceived
and in showing the company EXECUTIVES
that they can use their own research departments
to help solve
PRODUCTION and SALES and
DISTRIBUTION problems,
where they had not hitherto been used." *

And so,
let the skeptics who shun the
ABSTRACT
forever hold their peace,
and let them find out about
Lattice Theory which
they may not only find to be
"useful in their business" —
but will find also that
it throws so much light on
human psychology —
showing the human mind in action
other than the so-called
"universal drive to murder one's grandmother"
so popular with some
CLICHÉ - ADDICTS!

---

* Professor Morse kindly gave me permission to put this quotation in "short lines" to fit in with the format of all the books in this series.

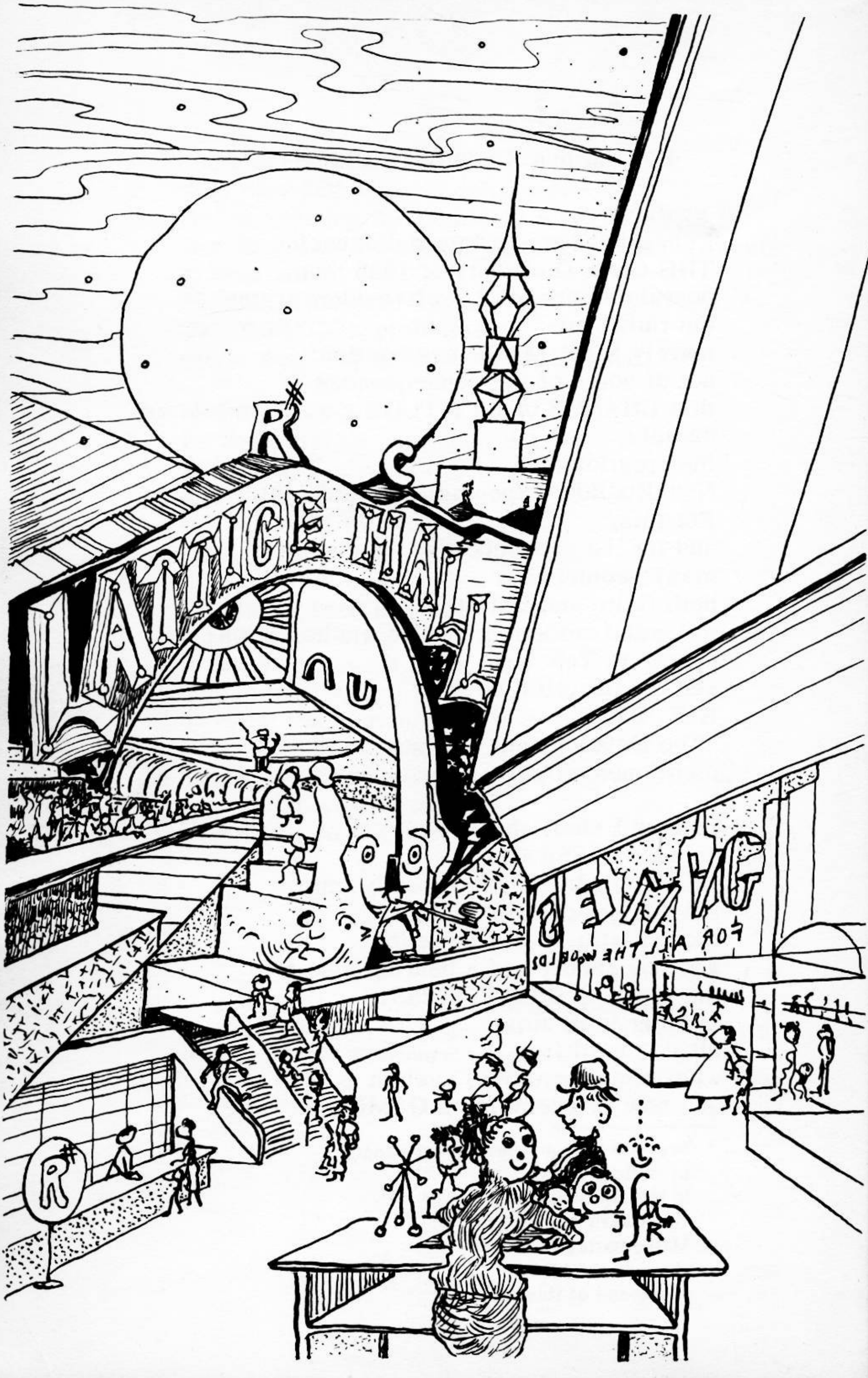
II. Various Kinds of ELEMENTS.

I shall not here go into a discussion of
THE Great Discovery of 1826 that
postulates are NOT "self-evident truths"
but rather are
merely MAN-MADE assumptions,
nor of some of the consequences of
this CHANGE of ATTITUDE toward postulates,
namely,
the creation of
Non-Euclidean geometries, etc., etc.
For this,
and for the subsequent development of
many geometries,
both finite and infinite,
the grand modern branch of mathematics
known as Topology,
see the beautiful book by
E.T. Bell:
"The Development of Mathematics", *
mentioned on page 1.

Instead I shall show you
some ALGEBRAS
which will lead more directly into
the subject of LATTICES;
though later you will see that
Lattice Theory has a bearing also on
the study of GEOMETRIES.
In order to do this,
let us recall first,
what a mathematical system is
and how it resembles a GAME.

---

* See also the various relevant papers,
by various authors,
in the list of publications of
The Galois Institute of
Mathematics and Art,
given in the bibliography
at the end of this book.

R#
C
ATTICE HAL
∩U
GAMES
FOR ALL THE WORLDS
R#
J R

I do not wish to imply that
a mathematical system is
"nothing but a game"
but that there are
certain aspects of a game
which may be helpful in
the consideration of
a mathematical system,
and everyone knows SOME game.
Thus,
we all know that

(a) a game must have some
EQUIPMENT —
something to play the game with;

(b) we must know what
we are supposed to DO with
the equipment;

(c) there must be RULES
according to which
the game is to be played.

Similarly,
a mathematical system also has:

(a) equipment,
called the ELEMENTS;

(b) OPERATIONS
to be performed upon the elements;

(c) basic POSTULATES (rules)
according to which
the operations are to be performed.

It is very helpful to realize this,
so that
whenever we encounter a new system —
say, a new algebra —

we immediately know enough to ask:

(a) what are the elements here?
(b) what are the operations?
(c) what are the postulates?

Knowing these three things
we then know
what game it is
that we are trying to play —
be it baseball or football or
bridge or pinnochle or
chess or checkers or
ordinary algebra or
some other "new-fangled" algebra,
or what have you.

Please do NOT think that
I am using the term
"new-fangled"
in a derogatory sense!
On the contrary
I wish to sell you the idea
(everybody is "selling" something
these days!)
that
some of these newer algebras
are MOST useful and practical * —
as well as
very simple and lots of fun.
Indeed that is the reason for this book
about Lattice Theory
which deals with
some of the most recent
powerful and fascinating algebras,
known as lattices.

---

* For example, Boolean Algebra,
for which see "Mits, Wits, and Logic",
another book in this series.
(See the bibliography at the end
of this book.)

But before going specifically into these,
let us take a look at

(a) various kinds of elements,
(b) various kinds of operations,
(c) various kinds of postulates,

which occur in
various kinds of algebras.

(a) Thus ELEMENTS may be:

1. NUMBERS — like

    (1) the set of ordinary whole numbers:
    (the "natural" numbers):
    1, 2, 3, ...
    or

    (2) the set of NEGATIVE whole numbers:
    $-1, -2, -3, \ldots\ldots$
    or

    (3) both positive and negative
    whole numbers, including zero:
    $\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots$
    or

    (4) the EVEN numbers ONLY
    including zero:
    $0, \pm 2, \pm 4, \pm 6, \ldots$
    or

    (5) the positive ODD numbers ONLY:
    1, 3, 5, ...
    or

    (6) ONLY the four numbers:
    0, 1, 2, 3,
    or

(7) other finite sets consisting of
only a few specified numbers,
instead of an infinite set
as in (1) to (5) above.

And of course we may include also
the fractions, obtaining

(8) the set of
RATIONAL * numbers (see p. 123);
or enlarge it further by
including the "Irrational" numbers,
thus obtaining

(9) the set of REAL * numbers.

And we can go even further and
include the "imaginary" numbers
obtaining

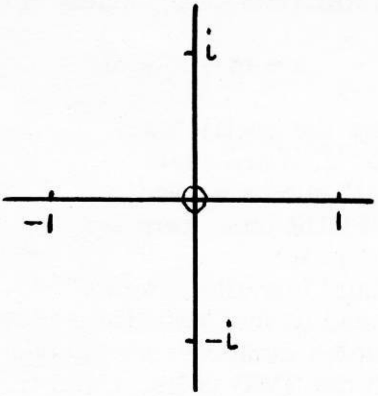
(10) the set of
COMPLEX * numbers,
which are numbers of the form
a + bi,
where a and b are "real" numbers,
and i is the "imaginary unit" *

$$\sqrt{-1},$$

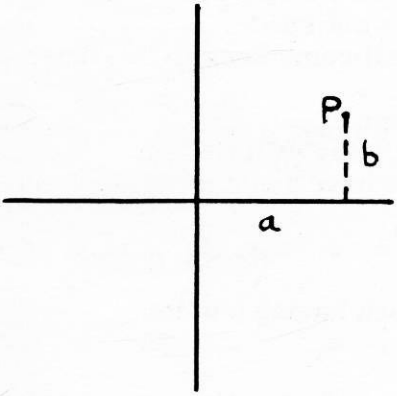
which is plotted in a direction
perpendicular to the
ordinary unit, 1, thus:

---

* For all these kinds of numbers,
see that excellent book:
"A Survey of Modern Algebra" by
Birkhoff and MacLane.



so that
a + bi
is represented by
the point, P:



Indeed, there are also
numbers known as

(11) HYPERCOMPLEX * numbers of the form

$$a + bi + cj + dk$$

where the coefficients
a, b, c, d are again
"real" numbers, and
the FOUR units here are
1, i, j, k,
giving "four-dimensional" points,
instead of the "two-dimensional"
complex numbers, mentioned above,
with the TWO units, 1 and i.

The Algebra of Hypercomplex numbers,
known as Quaternions
was developed by
Sir William Rowan Hamilton,
about 1835,
one of the first new algebras,
which had postulates which
seemed very bizarre at the time
but which we now take
quite in our stride.
We shall come back to this later.

And then,
generalizing still further,
we now have numbers of the form:

(12) $a_1e_1 + a_2e_2 + \dots + a_ne_n$,

each having n units,
$e_1, \quad e_2, \quad \dots, \quad e_n$

instead of only the four units
1, i, j, k, as in
Hamilton's hypercomplex numbers,
mentioned above.

---

* See footnote on page 11.

Now, mind you,
the quantity shown in (12)
represents of course,
for a given set of values of the a's,
a SINGLE "number",
and algebras which have such numbers
as their elements
are known as
LINEAR ALGEBRAS. *
More of these later also.

So you see that
the term "number"
has come a long way from
the familiar concept of
\$2 or even \$1, 000, 000.

Please do not ask every minute,
"What is the use of it?"
for, if you really wish to know,
you will possess your soul in patience,
and,
when you have finished this book,
you will have a glimpse into these
MODERN, HIGH-POWER
MATHEMATICAL "JETS",
and will appreciate not only
their tremendous PRACTICAL POWER,
but they will surely give you also
a great respect for the
HUMAN MIND
which can invent such things!

Hats off, please, to
"EGGHEADS",
without whom
the rest of us would be only
a bunch of mental pygmies!

---

* Since the expression in (12) is
of the first degree in the a's and hence
"linear".

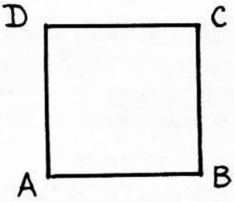
But this is not all —
for I must tell you that
the elements of a mathematical system
do NOT have to be NUMBERS at all,
not even the fancy kind mentioned above!

2. They may be:

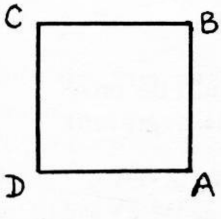
   (1) points, as in geometry;

   (2) substitutions
   (replacing some things by others!)
   as in Group Theory
   (see Chapter XIX.);

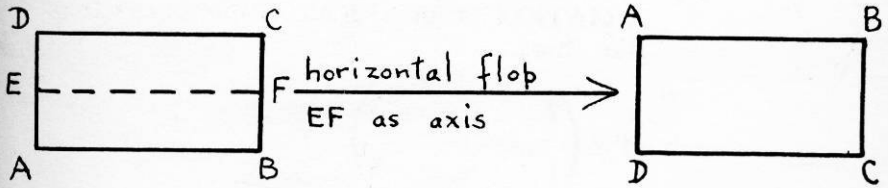
   (3) "revolutions"
   (not the political kind —
   but revolving a figure,
   like a square,
   through a certain angle) —
   for example, turning



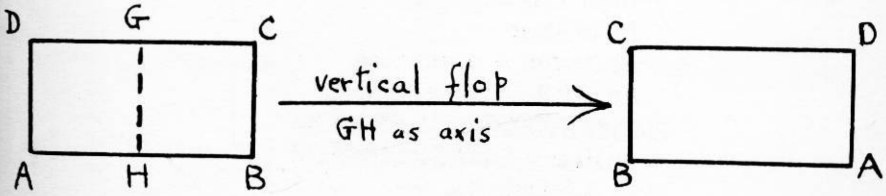
counterclockwise, through 90°,
so that it becomes



etc.
and
"flops"
(also not the political kind —
but flopping a figure,
like a rectangle,
over,
using some line as an axis),
thus:


D
C
E
F
A
B
horizontal flop
EF as axis
A
B
D
C

or


D
G
C
A
H
B
vertical flop
GH as axis
C
D
B
A

etc.

(4) events,
as in the
Einstein Theory of Relativity; *

(5) classes
as in logic; **

See * The Einstein Theory of Relativity,
** Mits, Wits, and Logic
both books in this series.
(See the bibliography at the end.)

(6) circles, spheres, and
many, many other
things or acts. *

Let us mention here
only one more kind of
element,
very important and interesting,
namely,

(7) MATRICES:
A MATRIX is an ARRAY of numbers
like this:

$$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$$

and it may have
more horizontal rows and
more vertical columns
than this one has.
Note that
although a matrix is
an ARRAY of numbers,
it is NOT itself a number,
unlike a DETERMINANT which
IS a NUMBER —
even though it also consists of
an array of numbers,
thus:

the DETERMINANT

$$\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \quad **$$

---

* See The Education of T.C. Mits, another book in this serie

** Note the STRAIGHT bars in the DETERMINANT, and
the CURVED ones in the MATRIX,
to distinguish one from the other.

represents the number
$(2x5) - (3x1) = 10 - 3 = 7$,
but not so the MATRIX

$$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$$

which may merely represent
the set of coefficients in

$$\left.\begin{array}{r} 2x + 3y \\ x + 5y \end{array}\right\}$$

You may wonder what can be done
with such arrays.
You will see later how they can be
"MULTIPLIED", etc.,
and you will no doubt
be interested to look up
the basic POSTULATES in
the Algebra of Matrices,
and to see how this Algebra
is connected with
LINEAR TRANSFORMATIONS
which are so important in
their applications to
various kinds of motions —
rotations, translations, etc. —
and various
stretchings, compressions, etc.,
so frequently met with in
the physical world.
You will enjoy looking further into
this fascinating Algebra
in the fine book by
Birkhoff and MacLane:
"A Survey of Modern Algebra",
mentioned on page 11.

And these are only SOME of
the kinds of ELEMENTS used in
Modern Algebras.

Thus you see the enormous expansion
that has taken place in the concept of
ELEMENTS of a mathematical system
in modern times.
This is a far cry from
the elementary notion that
mathematics deals only with ordinary
NUMBERS,
which some people think are
so "dry" and "boring"!
As a matter of fact,
you can now see that
almost anything can be used for
ELEMENTS,
and, if you have a problem,
you may not think that
it is a mathematical one,
and yet
it may actually be solvable by
these modern methods,
if you could learn to express yourself
in this modern and powerful language! *

It is definitely worth while for
EVERYONE
to look into this matter,
to see what it can do for you!

And when we come to
LATTICES,
you will find that
in some lattices
the elements themselves are
whole algebras!

---

* See Morse's reference to
"company executives",
page 6.

Does this begin to have
the modern flavor of
JETS and
SYNTHETIC EARTH-SATELLITES and
ATOMIC ENERGY?

IT DOES!

This is indeed
the mathematical counterpart of
these other
terrific modern accomplishments.
But
instead of being terrified by it,
it behooves us
to learn to think in a manner
commensurate with
these powerful "goings-on" —
and modern mathematics is
just the necessary thinking tool
that can help us adjust to all this
instead of cracking up under it
and resorting to
narcotics,
serving only to fill up
our hospitals and insane asylums.
Let us preserve our sanity
by adjusting to
the MODERN WORLD —
or else!
And an essential part of
this adjustment is
TO THINK STRAIGHT,
and there is nothing like
MODERN MATHEMATICS *
to stretch the imagination
and to conquer difficulties
OF ALL KINDS!

---

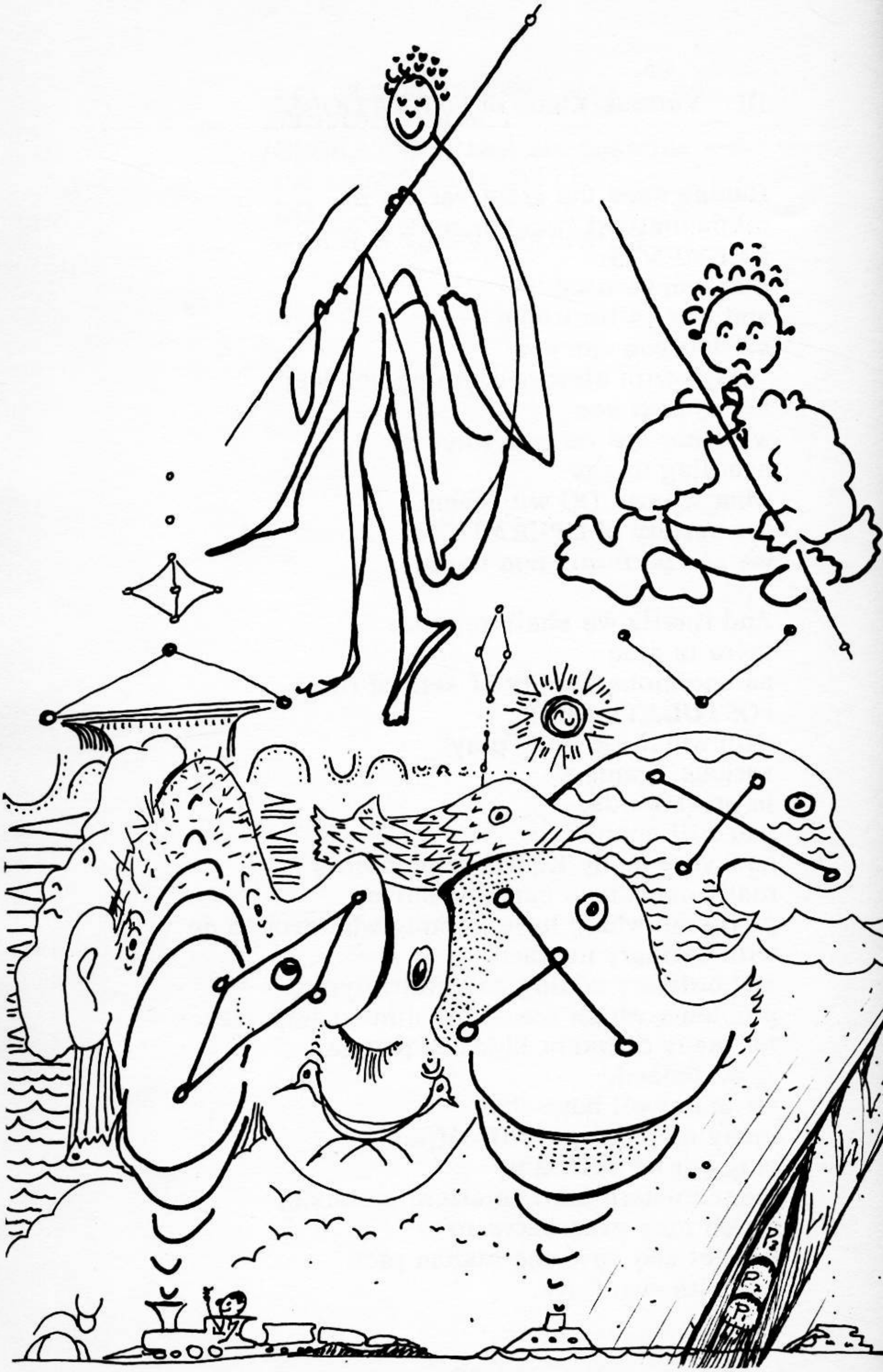
* There is a joke going around among
radio engineers that
we shall HAVE to learn
EITHER the language of
MATHEMATICS or RUSSIAN!

Let us next consider also
various kinds of
OPERATIONS that are possible —
the many things you can
DO
with this varied EQUIPMENT.

III. Various Kinds of OPERATIONS.

Having seen the great variety of
mathematical "equipment",
ELEMENTS,
that can be used —
and this is limitless —
so that you can now
"play" with almost anything you wish,
let us next see
what are the various ways of
handling things,
what we can DO with them,
the variety of OPERATIONS
we can perform upon them!

And finally we shall see that
there is also
an enormous variety of sets of rules,
POSTULATE SETS,
with which we can "play"
various "games".
In other words,
you will appreciate
how very many KINDS of problems
mathematicians can now solve,
problems which have nothing whatever to do
with ordinary numbers
and ordinary adding and multiplying, —
problems which are of the utmost importance
for many different kinds of people.
And, indeed,
we may well hope that
many more of our difficulties
may yet be solved by
these modern mathematical methods,
which may even serve to
protect and save the human race
from its own folly.


L_2
L_1
P_3
P_2
P_1

And so
let us learn how
to OPERATE
intelligently and effectively.

One of the OPERATIONS with which
you are already familiar from
ordinary arithmetic
is, of course, ADDITION,
as in

$$4 + 7 = 11.$$

And from ordinary algebra
you know also how to add
NEGATIVE numbers,
RATIONAL numbers,
REAL numbers, and
COMPLEX numbers.

It is important to point out here
that whereas we use the word
"ADDITION"
in connection with
ALL these kinds of numbers,
this operation is by no means the same
for these different kinds of numbers:
for instance —

$$4 + 7 = 11$$

but

$$(-4) + 7 = 3 :$$

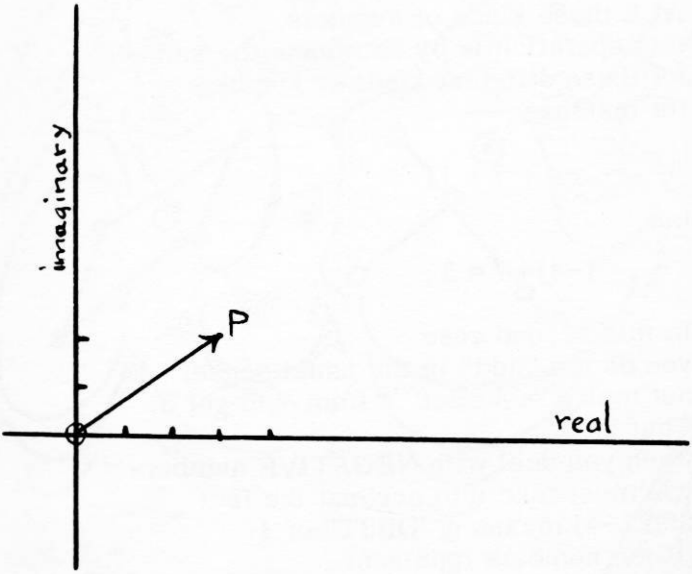
in this second case
you do not "add" in the usual sense,
but really "subtract" 4 from 7 to get 3.
That is,
when you deal with NEGATIVE numbers
you must take into account the fact
that (−4) means a "DEBT" of 4
(if the numbers represent
dollars, for instance)

so that the answer 3 is of course
most reasonable.
Thus, in this case,
"ADDITION" may be taken to mean
"balancing accounts",
that is,
taking into consideration
the "debits" and "credits"
instead of just
"adding" all the numbers indiscriminately.

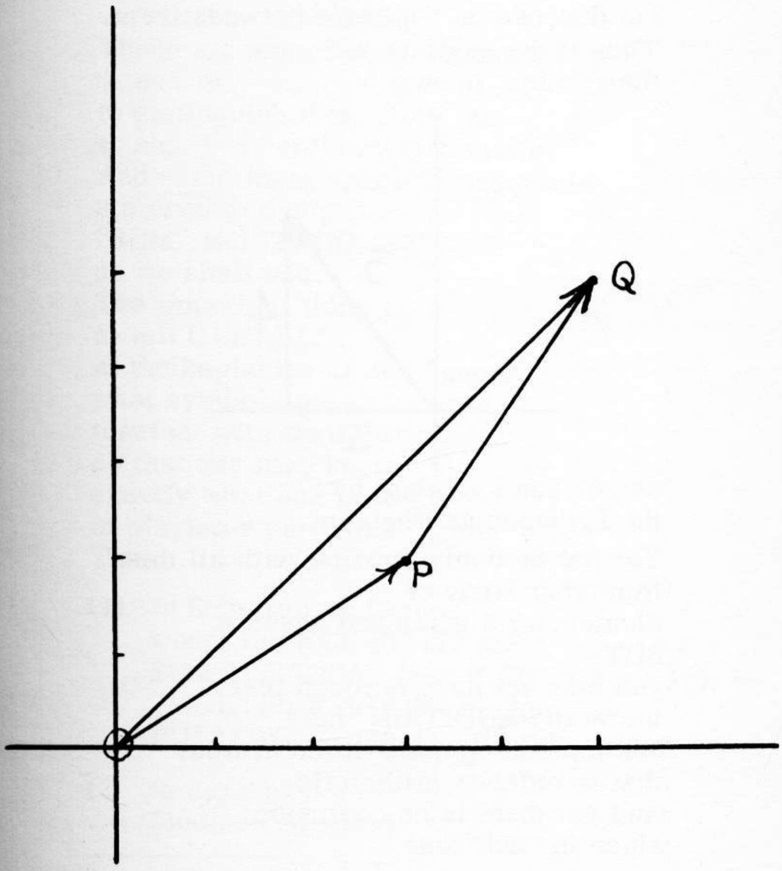
Similarly,
the meaning of "ADDITION" of
complex numbers
is again different,
thus:
consider the complex number

$$3 + 2i$$

which is represented graphically by



taking 3 units to the right on
the axis of "reals" and
2 units along the axis of
"imaginary numbers"
(see page (12);
then the complex number 3 + 2i
is represented by
either the point P, or the "vector" OP.*
Now, if you wish to "ADD" to this vector
another one, say, 2 + 3i,
you start this second vector at P
and go 2 units to the right and
3 up, landing at Q, thus:



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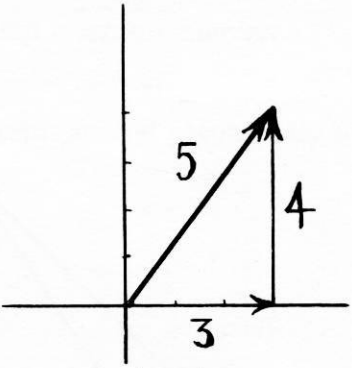
* A "vector" is a quantity which has
both magnitude and DIRECTION

Then $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ *
and of course OQ is shorter than
would be the sum of the
LENGTHS of OP and PQ
without taking into account their
DIRECTIONS,
as in ordinary addition.
Furthermore,
whereas the "ordinary" sum of
say, 3 and 4, is always 7,
the "vectorial" sum of 3 and 4
is NOT always the same
but depends on the angle between them.
Thus if the angle is 90°,
their "sum" is only 5:



as you can calculate by
the Pythagorean Theorem.

You are probably familiar with all this
from your study of
elementary algebra and physics,
BUT
you may not have realized that
the word "ADDITION" here
has a meaning quite different from
that in ordinary arithmetic.
And yet there is no confusion
since in each case

---

* The arrows indicate that
these lines represent "vectors".

the term is clearly defined
and you know exactly what to DO —
depending of course upon
the particular "game" which
you are playing.

Similarly
there are specific
but new and different meanings of
the words
"ADDITION" and "MULTIPLICATION"
in the various algebras which
we shall be discussing.
These are sometimes designated by
$\oplus$ and $\otimes$
to distinguish them from the
+ and x of ordinary arithmetic.
And sometimes still other symbols
are used to designate
"SUM" and "PRODUCT",
as we shall see.
The important thing is
to tell CLEARLY,
at the beginning of any "game",
what symbols are to be used,
together with their meanings,
so that one may know
exactly what one is to DO
in playing a particular "game".
Thus,

(1) in Substitution Groups,
where the ELEMENTS are
SUBSTITUTIONS (see 2.(2) on page 15),
"MULTIPLICATION" means
FOLLOWING one substitution by another.*

(2) in the Algebra of Classes
(Boolean Algebra), **

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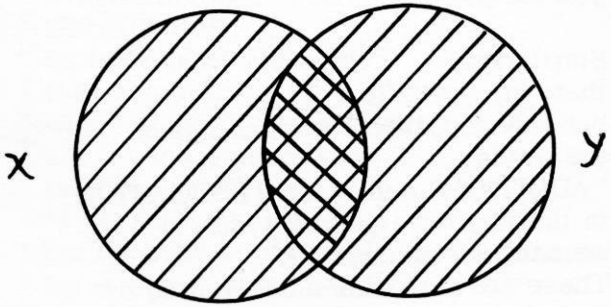
* See Chapter XIX.

** See Mits, Wits, and Logic,
another book in this series.

so useful in Logic,
"SUM" and "PRODUCT" of two classes
are defined as follows:



if the classes x and y are
represented by circles as shown here,
then $x \oplus y$ represents the
entire shaded area,
and $x \otimes y$ represents the
area common to both circles.

(3) in many simple and useful algebras
given by E.V. Huntington in his
"Fundamental Propositions of Algebra", *
$\oplus$ and $\otimes$ are given
many different meanings,
serving many appropriate purposes.

☐ (4) in the Theory of Matrices
there is still another use of the word
"MULTIPLICATION" —
To make this clear,
I must say something about
the Transformation of Axes,
with which you are probably
rather familiar from

---

* Published by
The Galois Institute of Mathematics and Art
(See Bibliography)

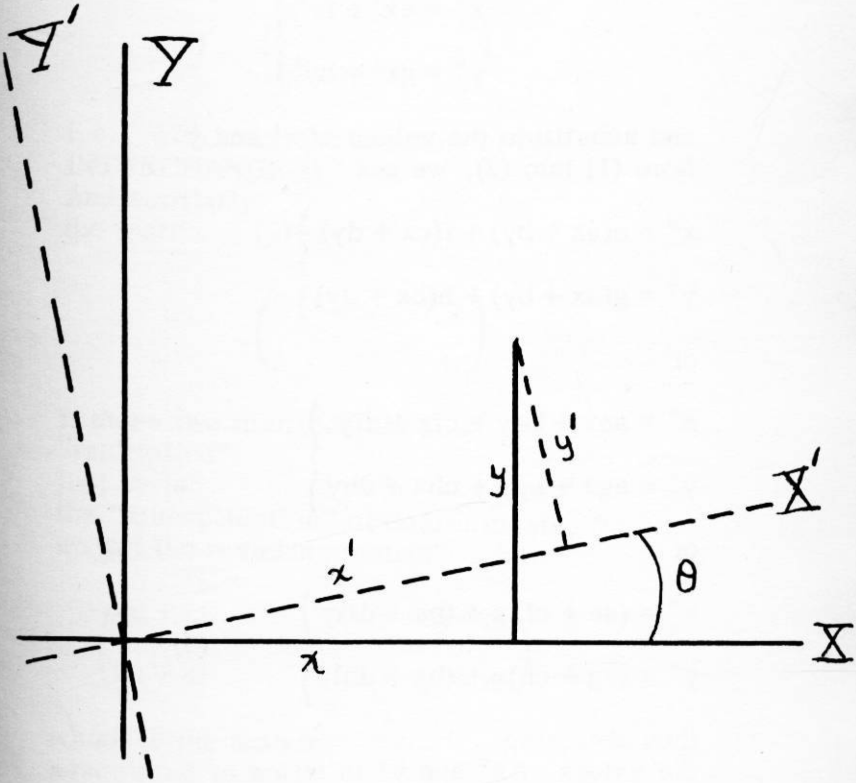
Analytic Geometry:
for example,
we often have occasion to
rotate a pair of rectangular axes,

X, Y

counter-clockwise to a new position

X', Y'

thus:



This transformation is represented by

$$\left.\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}\right\} \qquad (1)$$

where the coefficients a, b, c, d
are related to the angle, θ,
through which
the axes are rotated. *
If we now subject the axes to
a second rotation, represented by

$$\left.\begin{aligned} x'' &= ex' + fy' \\ y'' &= gx' + hy' \end{aligned}\right\} \qquad (2)$$

and substitute the values of x' and y'
from (1) into (2), we get

$$\left.\begin{aligned} x'' &= e(ax + by) + f(cx + dy) \\ y'' &= g(ax + by) + h(cx + dy) \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x'' &= aex + bey + cfx + dfy \\ y'' &= agx + bgy + chx + dhy \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x'' &= (ae + cf)x + (be + df)y \\ y'' &= (ag + ch)x + (bg + dh)y \end{aligned}\right\} \qquad (3)$$

thus obtaining
the values of x" and y" in terms of
the original x and y,
together with all the coefficients of
(1) and (2).

---

* See any book on Analytic Geometry.
Look up also its very interesting
application to Relativity Theory
in "Einstein's Theory of Relativity",
in this series
(See Bibliography).

Now, the final result, (3),
can be obtained much more EASILY
by "multiplying"
the MATRICES of the coefficients of
the two transformations (1) and (2)
as follows:

it is IMPORTANT to notice that
the matrix to be associated with (1) is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

i.e., with rows and columns of (1)
INTERCHANGED!
And similarly,
the matrix of (2) is

$$\begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

If these two matrices are now
"multiplied"
just as in
the "multiplication" of determinants, *
we get the resulting matrix

$$\begin{pmatrix} ae + cf & ag + ch \\ be + df & bg + dh \end{pmatrix}$$

which is the matrix
associated with (3)
(again with rows and columns interchanged!)

This interchange of rows and columns
is necessary,
as you can see from the details

---

* Look up "Determinants" in any Algebra book.

of the substitutions on page 30;
and of course
with this definition of "multiplication"
it is quite easy to
obtain the final result (3)
without going to all the trouble
of doing the detailed algebraic work
on page 30.

Indeed,
any given set of elements
may be used
to "play" VARIOUS "games"
(just as you can use a pack of cards
to play VARIOUS card-games),
depending on the meaning given to
$\oplus$ and $\otimes$:
thus,
if we use the set of real numbers
as elements,
we may define
"MULTIPLICATION"
in the ordinary way,
obtaining 3 x 7 = 21,
or
we may define "multiplication" thus:

$$x \otimes y = y$$

which would give

$$3 \otimes 7 = 7 \text{ and}$$

$$5 \otimes 2 = 2$$

etc.

The USE of such an algebra
is clearly given by
E.V. Huntington in
the little book mentioned above (p. 28).

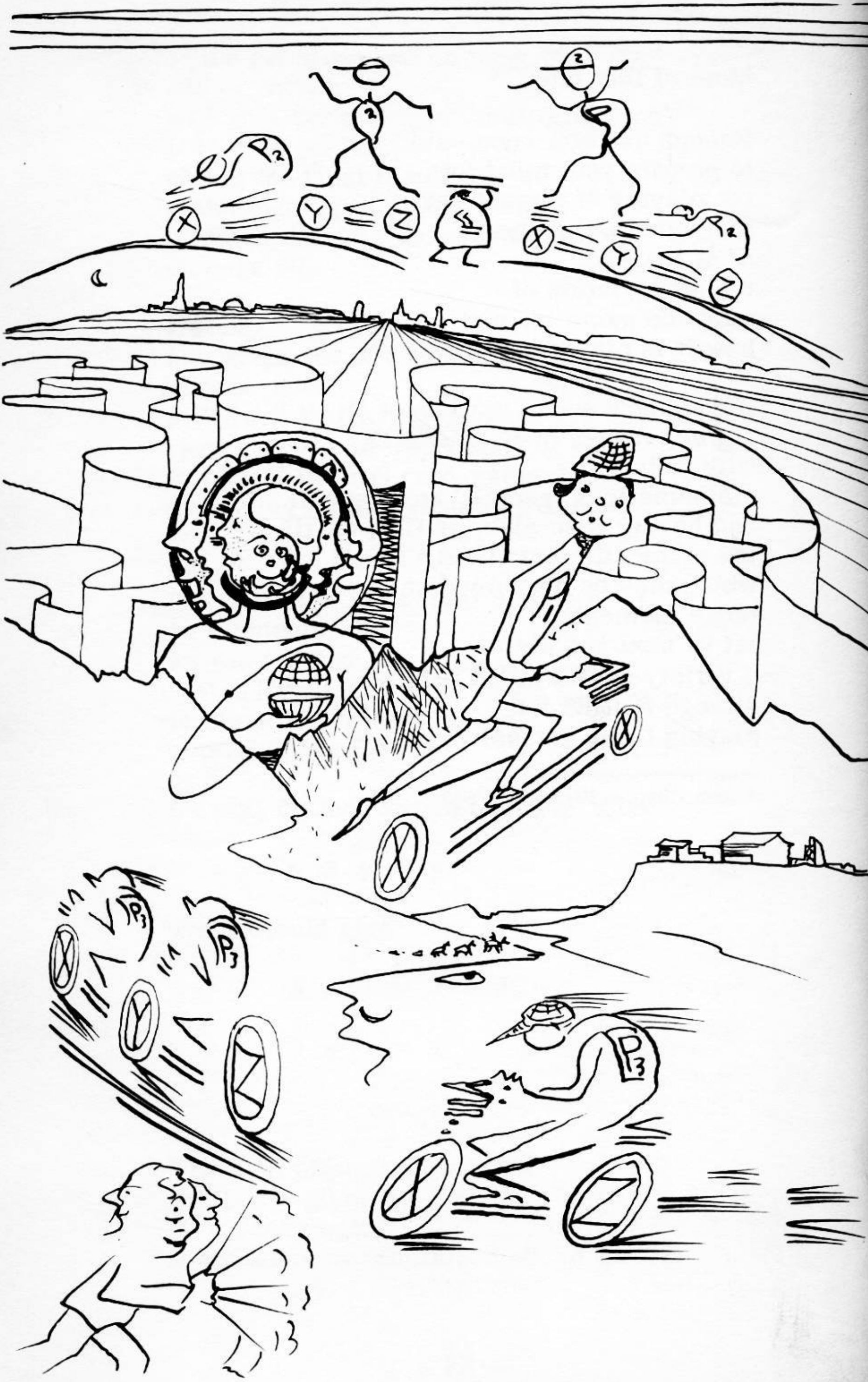
More of this later. *

Enough has now been said
to prepare your mind for
the playing of almost any
mathematical game,
or at least for
the appreciation of
what the game is, and
how it is played.

Having now seen, in Chapter II.,
a great variety of ELEMENTS
with which
mathematical "games" are played;
and having seen also, in Chapter III.,
the many different OPERATIONS
which may be performed on
these elements,
let us next see also
a variety of POSTULATES,
strange RULES for
playing these games!

---

* See Chapter IV; also p. 240.


P₂
X
2
Y ≤ Z
2
X ≤ Y ≤ Z
P₂
X
P₃
P₃
X
Y
Z
P₃
X ≤ Z

IV. Various Kinds of POSTULATES.

POSTULATES are, as you know,
the RULES for playing some "game".
Surely anyone in his right mind
would not even try to
play a game without knowing its
BASIC RULES!
And yet,
some people, young and old,
try to play the games of
arithmetic and algebra
WITHOUT EVER realizing
that these HAVE basic rules!
Now, mind you,
it is NOT BECAUSE these rules
are difficult,
NOR BECAUSE there are
too many of them!
On the contrary,
they are very simple
and very few,
as you will soon see.
Why is it then that
youngsters, in their study of
these subjects,
are usually NOT given
the POSTULATES?
But, instead, are given
various directions for
doing various little things, —
thousands of them! —
just as if a beginner in football
were told
"now grab the ball",
"now run this way",
"now run the other way",
etc., etc.,
without ever telling him
about the "goal",
or what he is really supposed to accomplish
or what he is allowed to do

or not allowed,
in short,
just pushing him around
in ways that may be clear enough to
the "pusher" or "teacher",
but which
the learner does not understand at all,
for he does not know what it is that
he is trying to do,
and gets quite bewildered by
the enormous number of details
with which he is overwhelmed!
Surely no one would ever think of
teaching football this way,
and yet this is the way
mathematics is often taught!
No wonder so many people
think they "hate math."

Now why is this so?

Is it because a certain psychologist
once emphasized the idea that
there are millions of
"S-R" bonds"
(Stimulus-Response bonds),
in arithmetic, for instance,
each of which is
something separate and distinct
and must be individually learned —
thus
the Stimulus "1 + 1"
must bring the Response "2",
"2 + 1" must bring "3",
etc., etc.,
ad infinitum.

Now I do not presume to
criticize this,
but surely it must be good psychology
to get a BROAD view of a game,
and be aware of
the set of rules which govern it!

Another objection sometimes raised to
this "postulational" approach
is that words like
"commutative",
"associative",
"distributive",
etc.,
(which describe some of the postulates,
as you will see) *
are just too hard for teenagers!
To which I can only say —
let us not underrate teenagers!
If we do not believe in them
and in their great drive to achieve,
we may turn them aside altogether
from good, hard pursuits,
and they may then use their strength
in other ways —
not necessarily good ways —
for strength they HAVE and
MUST use it.
And to think that
the above-mentioned words are
"too hard" for them
is just arrant nonsense!

Before giving you
a few illustrations of
well-known and important algebras
which will then lead into
Lattice Theory,
as promised,
let me first mention briefly
some individual postulates which
will show the VARIETY of
RULES that are possible in
these different algebras;
for instance:

(1) If you take the set of elements
in (4) on page 10 —

---

* See pages 38ff.

the set of EVEN numbers, including 0,
and if you take
ordinary addition as the operation,
you can see that
the postulate of "closure" holds here,
i.e., if you add any two
of the elements,
the result is also
an element of this set,
since the sum of
any two EVEN numbers
is itself an EVEN number.
Whereas the set (5) on page 10 —
the positive ODD numbers only —
obviously does NOT have
closure under addition,
since the sum of any two ODD numbers
is NOT itself an ODD number,
but is an EVEN one
(which is of course
NOT included in the given set):
thus $1 + 3 = 4$, $7 + 11 = 18$, etc.
Similarly,
if the given set of elements
consists only of the five numbers

$$1, \ -1, \ 0, \ i, \ -i$$

(where i is the $\sqrt{-1}$)
then this set does NOT have
closure under addition,
since, for example,
$1 + 1 = 2$
and 2 is NOT in the given set.
But this same set of numbers
DOES have closure under MULTIPLICATION —
try it and see!
(Remember that $i \times i = -1$, etc.)

Thus some systems
do have CLOSURE under

some operation,
and others do not.

CLOSURE is a very important property,
since it guarantees first of all that
you will get an answer at all,
after operating on the elements!
Thus
one of the basic properties of
a Group *
is Closure;
and ordinary arithmetic and algebra
have closure under
both addition and multiplication,
and so on.

(2) Take next the postulate

$$a \otimes b = b \otimes a,$$

known as the
Commutative Law of Multiplication,
which says of course that
if you multiply two elements, a and b,
it does not matter which one
you take first, —
the answer will be the same
in either case.
Now, some systems
do satisfy this postulate
and some do not —
for instance,
consider a system in which
$\otimes$ is defined by
$a \otimes b = b$, as on page 32:
here, then, $b \otimes a = a$,
and consequently

$$a \otimes b \neq b \otimes a,$$

---

* See page 48

i.e. multiplication here is
NOT commutative.
Whereas, in ordinary arithmetic
the commutative law of multiplication
DOES hold —

i.e. $5 \times 2 = 10$ and
$2 \times 5 = 10$

the answer is 10 regardless of
the order of the factors.
We are so ACCUSTOMED to this in
ordinary arithmetic and algebra,
that it seems unimportant to
mention it —
though when you realize that
some systems do NOT have
this postulate,
you appreciate the need for
calling attention to it.
Thus, in general,
"multiplication" in a Group
is NOT commutative —
this property of commutativity is
NOT one of the basic postulates for a Group; *
but when it IS included as
a postulate,
the system is still a Group,
but of a special kind,
and is called
a Commutative or Abelian ** Group.

(3) Another postulate known as
the Associative Law for a
given operation $\oplus$
is expressed by

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

---

* See pages 48, 49.

** Named for the mathematician, Abel.

This of course holds in arithmetic,
thus, if we wish to find
the SUM of THREE numbers,
we may "associate" them in
two different ways,
as shown above,
and still get the same answer:

$7 + 4 + 2 = (7 + 4) + 2 = 11 + 2 = 13$

or, also,

$7 + 4 + 2 = 7 + (4 + 2) = 7 + 6 = 13.$

This is a most important postulate,
it holds in many systems,
it is one of the
basic requirements for
a system to be a Group; *
it holds for
BOTH Addition and Multiplication in
ordinary algebra,
as well as in Boolean Algebra,
etc.

And yet it too is
not inevitable —
it is possible to have a system
without it (see below),
which again emphasizes the fact that
one must explicitly mention
what are the postulates of a system,
which characterize it,
just as stating the rules of
a particular card game
identifies the game.

In the little book mentioned on p. 28,
Huntington gives an algebra in which
this associative law

---

* See page 49.

does NOT hold;
in it he defines $\oplus$ as follows:

$$x \oplus y = \frac{x+y}{3}$$

where the ordinary + of course
means ordinary addition;
thus

$$5 \oplus 4 = \frac{5+4}{3} = \frac{9}{3} = 3.$$

Now, if we wish to "add"
THREE numbers here

$$5 \oplus 4 \oplus 2$$

we find that
we do NOT get the SAME answer
when we associate them in
the two different ways shown on p. 39:
thus,

$$(5 \oplus 4) \oplus 2 = \frac{5+4}{3} \oplus 2 = \frac{9}{3} \oplus 2$$

$$= 3 \oplus 2 = \frac{3+2}{3} = \frac{5}{3},$$

whereas

$$5 \oplus (4 \oplus 2) = 5 \oplus \frac{4+2}{3} =$$

$$5 \oplus \frac{6}{3} = 5 \oplus 2 = \frac{5+2}{3} = \frac{7}{3};$$

that is,
by associating the numbers in
the two different ways, we get
two DIFFERENT answers here —
so that this operation $\oplus$ is
NOT associative.

(4) Let me mention only
one more postulate
which is also very well-known —
the "Distributive Law".
You are familiar with it in
elementary algebra,
in which

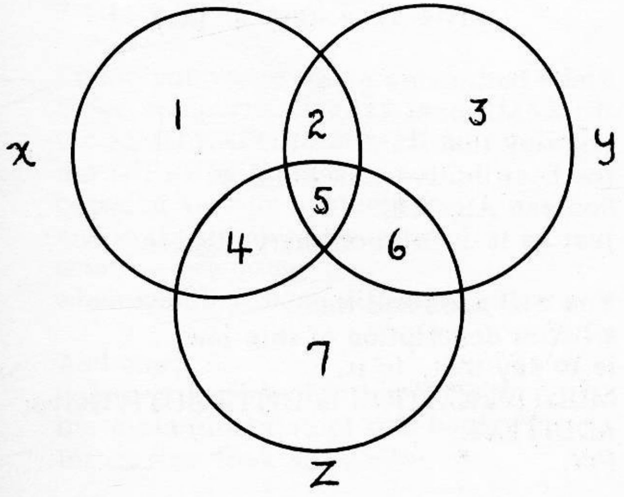
$x(y + z) = (xy) + (xz)$,

showing that x is "distributed" to
both y and z.
Thus, for example,

$7 \times (4 + 2) = (7 \times 4) + (7 \times 2)$

i.e.,
the left side gives 7 x 6 = 42, and
the right side gives 28 + 14 = 42,
the SAME answer.

It is interesting to note that
this postulate holds also in
Boolean Algebra
(the Algebra of Classes);
consider the three classes, x, y, z:



and designate the various areas
by the numbers from 1 to 7,
as shown.

Then we may designate the area $(y + z)$
by 234567
(see the definition of
the SUM of two classes, on p. 28);
and $x = 1245$, does it not?
Then the product

$$x(y + z),$$

which is the part
COMMON to these two classes (p. 28),
is the area 245, is it not?
Now let us find

$$(xy) + (xz);$$

(xy) is the part common to x and y,
that is, it is the area 25,
and (xz) is 45;
then their sum is 245, is it not?
Thus we find that

$$x(y + z) = (xy) + (xz)$$

since both sides give
the SAME area 245,
showing that
the Distributive Law holds in
Boolean Algebra,
just as it does in ordinary algebra.

You will soon see that
a better description of this law
is to say that, in it,
MULTIPLICATION is DISTRIBUTIVE over
ADDITION.
For,

in Boolean Algebra,
there is ANOTHER Distributive Law,
in which
ADDITION is DISTRIBUTIVE over
MULTIPLICATION
thus:

$x + (yz) = (x + y)(x + z)$.

You can easily see that
this holds for three classes,
by proceeding as we did above.
But, you can also easily see that
this second Distributive Law
does NOT hold in
ordinary algebra and arithmetic:
for example:

$7 + (4 \times 2) = 7 + 8 = 15$

whereas

$(7 + 4)(7 + 2) = 11 \times 9 = 99$,

and therefore

$7 + (4 \times 2) \neq (7 + 4)(7 + 2)$!

I hope you agree that
these comparisons emphasize
the SIMILARITIES as well as
the DIFFERENCES
between various systems,
and make us more aware of
what we are doing
when we play these different games.

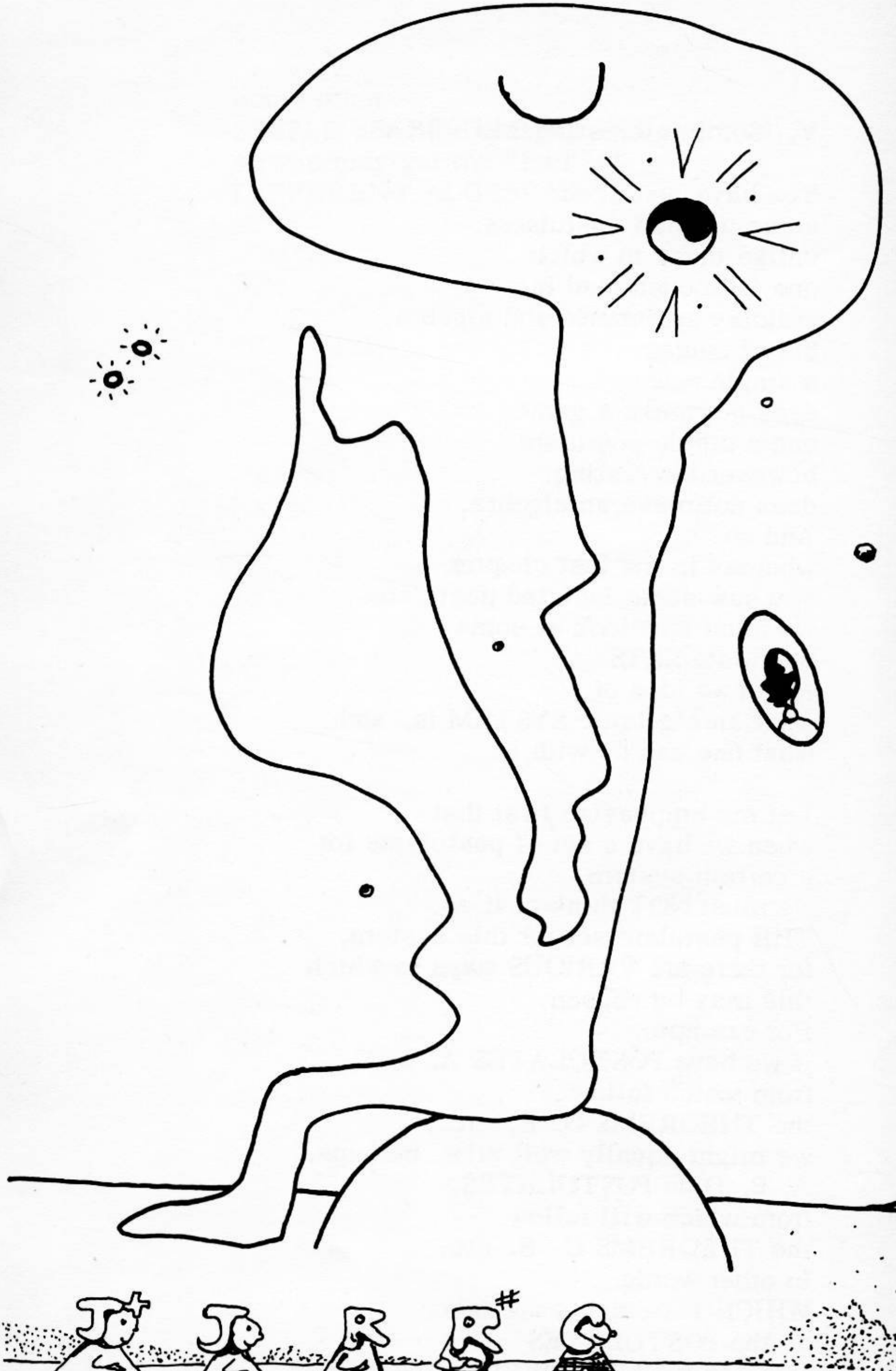
And now,
before going into Lattice Theory,
the main objective of this book,
let us first look at

some other
SIMPLE but very IMPORTANT systems,
so you may get the "feel" of
DIFFERENT ALGEBRAS.

V. Some interesting ALGEBRAS.

You have just seen
some unusual postulates,
unlike those to which
one is accustomed in
ordinary arithmetic and algebra.
But of course
a single rule
does not make a game,
and a single postulate,
however interesting,
does not make an algebra.
And so
whereas in the last chapter
you saw some isolated postulates,
we must now look at some
postulate SETS
to get an idea of
what an algebraic SYSTEM is, and
what one can do with it.

Let me emphasize first that
when we have a set of postulates for
a certain system,
we must NOT think of it as
THE postulate set for this system,
for there are VARIOUS ways in which
this may be chosen.
For example,
if we have POSTULATES A, B, C
from which follow
the THEOREMS D, E, etc.,
we might equally well take, perhaps,
A, B, D as POSTULATES,
from which will follow
the THEOREMS C, E, etc.
In other words,
WHICH statements are taken
as the POSTULATES
(i.e. as the UNPROVED propositions)


J
J
R
R
C

and which are then PROVED from these,
may depend on various considerations of
simplicity or other objectives!
Thus
A postulate set for Euclidean Geometry
is given by O. Veblen in his
"The Foundations of Geometry"; *
a modified form of it
which looks very different but is
equivalent to it,
is given by A. Church in
"The Euclidean Parallel Postulate"; **
and any GOOD high school text-book on
Euclidean Geometry has also
an EQUIVALENT set —
among these I wish to make special mention of
an excellent elementary text-book called
"Basic Geometry" by
Birkhoff † and Beatley,
in which the authors give
only FIVE basic principles or POSTULATES for
Euclidean plane geometry,
one of which is:
"Two triangles are similar if
an angle of one equals an angle of the other
and the sides including these angles are proportional."
Now every high school boy knows that
this is USUALLY NOT taken as a POSTULATE,
but is one of the THEOREMS in "Book III" of
most geometry text-books —
but by taking it as a POSTULATE
together with four other very simple ones;
Birkhoff and Beatley have written a text which is
simple, though rigorous,
and which SHOULD be used by

---

* One of the excellent articles in
"Monographs on Modern Mathematics"
(See Bibliography).

** See Bibliography

† This is George D. Birkhoff,
a very celebrated mathematician,
father of Garrett Birkhoff whose name is
associated with Lattice Theory
the principal subject-matter of this little book.

every high school student of elementary Geometry!

And so,
bearing in mind that
we must always speak of
A and not THE set of postulates for
any system,
let me now give you a few samples:

(1) I shall not repeat here
the SET of POSTULATES for
Boolean Algebra,
with its marvellous applications
to Logic,
to electrical circuit theory,
etc., etc.,
which I have given elsewhere. **
But we shall look again at
Boolean Algebras
(Note the plural!)
from the viewpoint of
Lattice Theory
in Chapter XVIII.

(2) A Group is another system,
an algebra,
very SIMPLE,
and extremely USEFUL.
I shall say here only that
a Group has ONLY ONE operation —
let us represent it by o —
and that we may take for it
the following FOUR postulates:
If we represent the elements by
a, b, c, ...
then a set of
four postulates for a Group
may be expressed as follows:

(a) Closure (see p. 37);

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* This book has been recently reprinted by the Chelsea Publishing Co., New York

** See "Mits, Wits, and Logic"
in this series
(See Bibliography)

(b) (a o b) o c = a o (b o c)
i.e., the associative law holds
for the operation o; (p. 39);

(c) there is an
"Identity" element, I,
such that
$$a \circ I = a$$
for any element, a.
Compare the Identity element,
zero,
for ordinary addition,
where $a + 0 = a$;
and the Identity element, 1,
for ordinary multiplication,
where $a \times 1 = a$.
Note that
an Identity element always
leaves an element, a,
UNCHANGED.
This will be clearer in
the following illustrations;

(d) every element, a, has
an "inverse" element, a',
such that
$a \circ a' = I$ (the identity).
Thus, under ordinary addition,
the "inverse" of 3 is −3,
since $3 + (-3) = 0$,
where 0 is the identity element.
Similarly,
the "multiplicative inverse" of 3
is 1/3 since
$3 \times (1/3) = 1$,
where 1 is the
identity element for multiplication
(see above),
etc.
Thus an "inverse" of an element
depends of course upon
the particular operation in question.
Furthermore,
as stated on pages 46ff,
a set of postulates for any system
is not unique,

but may be replaced by
an "equivalent" set.
Thus, for a Group,
instead of the four postulates given above,
we may take alternative sets,
as shown in Appendix I., p. 267.

Let us now look at
a few simple illustrations of
systems
and test them to see whether
they are groups or not,
using the set of four postulates
given on pages 48, 49.

1. Take the set of elements
   consisting of all the
   positive integers;
   this set is often designated by

   $$J^{+}.$$

   And let the one operation be
   ordinary addition.
   Is this system a group?

   Well,
   let us see if it satisfies
   the four postulates for a group (p. 48):

   (a) it does have "closure",
   since the sum of
   any two positive integers
   is itself a positive integer;

   (b) it DOES obey the
   associative law
   (just as in ordinary arithmetic);

   (c) but it does NOT have an
   identity element,
   since there is
   NO positive integer which
   when added to another
   will leave that one unchanged.
   Note that zero, 0, is
   needed for this

under addition (p. 49)
and there is no 0 in J^+.

And so J^+ under addition is
NOT a group.

2. Now how about J^+ under multiplication?
Here (a), (b), AND (c)
ARE satisfied
(the identity element now is 1 — p. 49
and 1 IS present in J^+),
BUT
(d) is NOT satisfied since there are
no multiplicative inverses in J^+
(thus 1/3, for instance,
is not in J^+
which consists of INTEGERS ONLY).
So this system is also
NOT a group.

3. On the other hand,
if we take
not only the POSITIVE integers,
but ALL the integers,
POSITIVE and NEGATIVE
AND INCLUDE ALSO ZERO,
thus,
...., −3, −2, −1, 0, 1, 2, 3, ...
(this set is often designated by J)
and take ADDITION as the operation,
then this system
IS a group,
since it satisfies
ALL four requirements on p. 48,
does it not?

4. Similarly
the set of positive rational numbers
is a group under MULTIPLICATION —
test it and see.

And so on.

Thus
it is quite easy to tell
whether or not
a system is a group.
The amazing thing is

how much has been done
with this simple instrument!

Galois,
the great, young mathematician,
used it with tremendous skill —
proving, among other things,
that the general algebraic equation
of degree greater than 4
CANNOT be "solved by radicals", *
proving also that
the general angle
CANNOT be TRISECTED by means of
straight edge and compasses. *
Indeed,
as E.T. Bell says: **
"Wherever groups disclosed themselves,
or could be introduced,
simplicity and harmony
crystallized out of comparative chaos.

The idea of a group
was one of the outstanding additions
to the apparatus of scientific thought
of the last century."

For more about groups and
their connection with
Lattices,
see Chapter XIX.

Let us next look briefly at
some other simple algebras:

(3) An INTEGRAL DOMAIN:
this system has
TWO operations, + and x,
and a POSTULATE set as follows:

---

* Just what this means, and how it is proved
is a fascinating story:
See "Galois and the Theory of Groups"
in this series.

** In "The Queen of the Sciences"

(a) Closure for +.
(b) $(a + b) + c = a + (b + c)$
(c) $a + 0 = a$
(d) $a + a' = 0$ $(a' = -a)$
(e) $a + b = b + a$
(f) Closure for x
(g) $(ab)c = a(bc)$
(h) $a \times 1 = 1 \times a = a$
(i) $ab = ba$
(j) $a(b + c) = ab + ac$
(k) If $c \neq 0$ and $ca = cb$,
then $a = b$.

If you look at these postulates
closely,
you will see that
the first five, from (a) to (e)
make this system
an Abelian Group for + (see pp. 39 and 48).
Note, however, that
there is NO MULTIPLICATIVE INVERSE,
i.e., there is no a' such that
$a \times a' = 1$ (p. 49),
so this system is NOT a group for x.
Further,
it obeys the "distributive law" (j) *
and also
what is known as
the "cancellation law" (k), above,
which grants the right to cancel the c in

$$ca = cb$$

(if $c \neq 0$ of course),
and obtain $a = b$.
This is undoubtedly so familiar to you
from ordinary algebra,
that you probably take this right
for granted,
and perhaps think that
the cancellation law

---

* p. 42.

must necessarily hold in all systems!
Having already met with situations in which
ab does NOT equal ba (p. 38),
or in which
the associative law does NOT hold (p. 41),
etc., etc.,
you may be sophisticated enough by now
to realize that
all these postulates are
MAN-MADE,
that there is nothing SACRED about them,
that you can make up systems in which
they may or may not hold, —
it is simply up to you
to make up systems which are
CONSISTENT
(i.e., your rules or postulates
must NOT contradict each other —
but otherwise you are quite free to choose!)
and USEFUL,
if you are clever enough to do so!

You will soon see
an illustration in which
the cancellation law (k)
does NOT hold!

But first let me say that
if to the eleven postulates of
an INTEGRAL DOMAIN (p. 53)
you add a twelfth one,
namely,

$a' \times a = a \times a' = 1$ (where $a' = 1/a$),

for every a, (not zero)
that is, if you introduce
a multiplicative inverse for every a,
then

(4) the system is called a
FIELD.
Thus a Field is
an Abelian Group for BOTH + and x, (omitting 0 element)
and it obeys the distributive law (j).

You will be interested to know that
it is not necessary to include (k) here
for this now follows automatically from
the remaining postulates! *

(5) Further:
If from the postulate set for
an INTEGRAL DOMAIN
you omit (h), (i), and (k),
the system becomes a
RING.
And

(6) if (i) DOES hold BUT
not necessarily (h) and (k),
then the system is a
COMMUTATIVE RING.
And

(7) if (h) also holds,
it is a
COMMUTATIVE RING WITH UNITY.

etc., etc.

Thus you see that a great many
DIFFERENT ALGEBRAS
are possible,
and of course this is
only the beginning!
To get some idea of
the really great variety of algebras,
see "A Survey of Modern Algebra" by
Birkhoff and MacLane,
which has already been referred to
several times
above.

For the moment
you may be interested in
testing the following systems,
to see whether they satisfy
the appropriate postulate set:

---

* See Birkhoff and MacLane:
"Survey of Modern Algebra".

With the operations + and x
show that

(1) J (the set of positive and negative integers, and zero)
is an INTEGRAL DOMAIN
but NOT a FIELD. *

(2) R (the set of positive and negative rational numbers, and zero)
is a FIELD.

(3) $R^{\#}$ (the set of positive and negative REAL numbers, and zero)
is a FIELD.

(4) The set of numbers of the form

$$x + y\sqrt{2}$$

(where x and y are integers)
is an INTEGRAL DOMAIN
but NOT a FIELD.
But if x and y are rational numbers,
then this system IS a FIELD. **

In the next chapter
you will be amused to see
a system in which
the cancellation law (k)
does NOT hold.

---

* See pages 52-54.

** Show that if $a = x + y\sqrt{2}$, then there is an $a' = 1/(x + y\sqrt{2})$ which is still of the given form $A + B\sqrt{2}$, where A and B are both rational numbers.
Do this by "rationalizing" the denominator of a',
as is done in elementary algebra.

VI. An Engaging Ring.

To illustrate this,
let me start with something which
is very familiar:

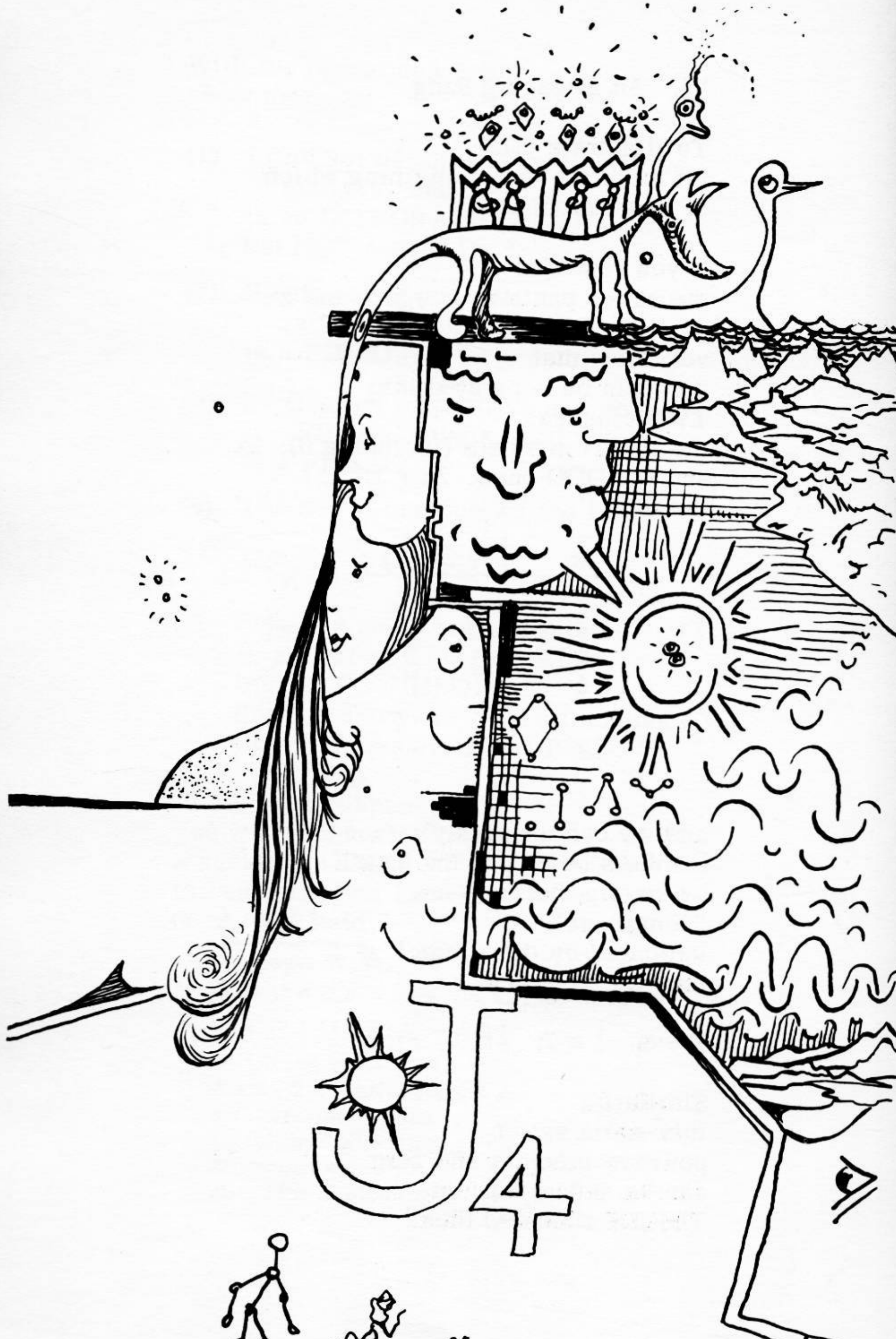
If you consider
the set of positive integers and zero,
0, 1, 2, 3, ...
you know that
they can be separated into
TWO classes:
the EVEN numbers (including 0), 2x,
and the ODD ones, 2x + 1,
thus:

$2x$	$2x + 1$
0	1
2	3
4	5
6	7
.	.
.	.
.	.

and we can express the fact that
numbers belong to the SAME class
by saying that they are
"congruent",
indicated by the symbol $\equiv$;
thus we have here

$2 \equiv 6$, $1 \equiv 7$, etc., etc.

Similarly,
this same set of
positive integers and zero
can be separated into
THREE classes, thus:


J4

$3x$	$3x + 1$	$3x + 2$
0	1	2
3	4	5
6	7	8
9	10	11
.	.	.
.	.	.
.	.	.

where the numbers in the first column
are all multiples of 3,
those in the second column are each
1 more than some multiple of 3
(thus $7 = (2x3) + 1$, $10 = (3x3) + 1$, etc.),
and those in the last column are each
2 more than some multiple of 3,
(thus $5 = (1x3) + 2$, $8 = (2x3) + 2$, etc.).
And, again, numbers in the
same class or column
are "congruent", $\equiv$,
thus $2 \equiv 5$, etc.

Now, since in the
TWO-class arrangement on p. 57
2 is NOT $\equiv 5$,
whereas here in the
THREE-class arrangement
2 IS $\equiv 5$,
we must obviously INDICATE
the particular class arrangement
in question,
which is done by saying that
in the first case
the "modulus" is 2,
and, in the second case,
the "modulus" is 3;
and so we say that
$2 \equiv 6$ "modulo" 2
("modulo" 2 means "with the modulus" 2)
or, for short,

$$2 \equiv 6 \pmod{2}.$$

And similarly

$$2 \equiv 5 \text{ (mod. 3)}.$$

And of course
the set of integers and zero
may be arranged in
ANY number of classes,
using ANY integer as modulus.

Now, let us go back to
the case of TWO classes:
since there all EVEN numbers
are "congruent" to 0,
and all ODD ones
are congruent to 1,
the numbers 0 and 1
are really all we need to
represent ALL the integers modulo 2.
Thus, the following
"Addition and Multiplication Tables"
are sufficient for this system:

⊕	0	1
0	0	1
1	1	0

⊗	0	1
0	0	0
1	0	1

That is to say,
since 0 here really means
ANY EVEN NUMBER,
and 1 means
ANY ODD NUMBER,
these tables really say that

(1) if any EVEN number (0)
is added to another EVEN number (0)
the SUM is again an EVEN number (0),

$$\text{thus } 0 + 0 = 0$$

as shown in the Addition Table.

And the SUM of an EVEN number (0)
and an ODD number (1)
gives an ODD number,
i.e., $0 + 1 = 1$
and $1 + 0 = 1$.

Finally,
the SUM of two ODD numbers (1)
is an EVEN number, (0)

thus $1 + 1 = 0$.

(2) Similarly for
the PRODUCT of
EVEN and ODD numbers —
EVEN x EVEN = EVEN ($0 \times 0 = 0$)
(like $4 \times 6 = 24$),
EVEN x ODD = EVEN ($0 \times 1 = 0$)
(like $4 \times 5 = 20$),
ODD x ODD = ODD ($1 \times 1 = 1$)
(like $3 \times 5 = 15$).

And all this,
using 0 for all even numbers,
and 1 for all odd numbers,
as explained above,
is summarized in
the two simple tables on p. 59,
one for ADDITION and
the other for MULTIPLICATION.
This system is briefly designated by

$$J_2$$

Similarly we may have J_n
for any modulus, n.
Thus,
if we arrange the integers (and zero)
in FOUR classes, thus:

4x	4x + 1	4x +2	4x + 3
0	1	2	3
4	5	6	7
8	9	10	11
.	.	.	.
.	.	.	.
.	.	.	.

we may use 0, 1, 2, 3
to represent the four classes,
since every integer is
"congruent" to
one of these four, modulo 4.
We would then have,
for this system, J_4 ,
the following
ADDITION and MULTIPLICATION TABLES:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

You realize that the numbers
are added and multiplied
in the usual manner,
except that
whenever the result is
greater than 3,
you represent it by
one of the four numbers:

0, 1, 2, 3

to which the result is congruent.

Thus,

$$3 \times 3 = 9$$

but $9 \equiv 1 \pmod{4}$
therefore $3 \times 3 = 1$ here
as shown in the Multiplication Table, p. 61.
And so on for other numbers.

Now observe that
the cancellation law, **(k)**, on p. 53
does NOT always work here:
for example,

$2 \times 1 = 2$ and $2 \times 3 = 2$,

therefore $2 \times 1 = 2 \times 3$.
And yet,
if you now cancelled the 2,
you would get $1 = 3$,
which is FALSE!
Thus it is NOT legitimate
to cancel the 2,
i.e., the cancellation law
does NOT work here. *

But since
all the other postulates **(a)** - **(j)**, p. 53,
DO hold here,

---

* An interesting theorem here is:
The cancellation law for multiplication
HOLDS in the system J_n
(of integers modulo n)
IF and ONLY IF
n is PRIME.
In the above illustration
$n = 4$, which is NOT prime,
and therefore
the cancellation law did NOT hold there,
as you saw.
But try it for $n = 5$ (which is prime)
and show that, in that case,
the cancellation law does hold,
as required by the theorem stated above.

this system, J_4 ,
is therefore a
COMMUTATIVE RING WITH UNITY (p. 55).

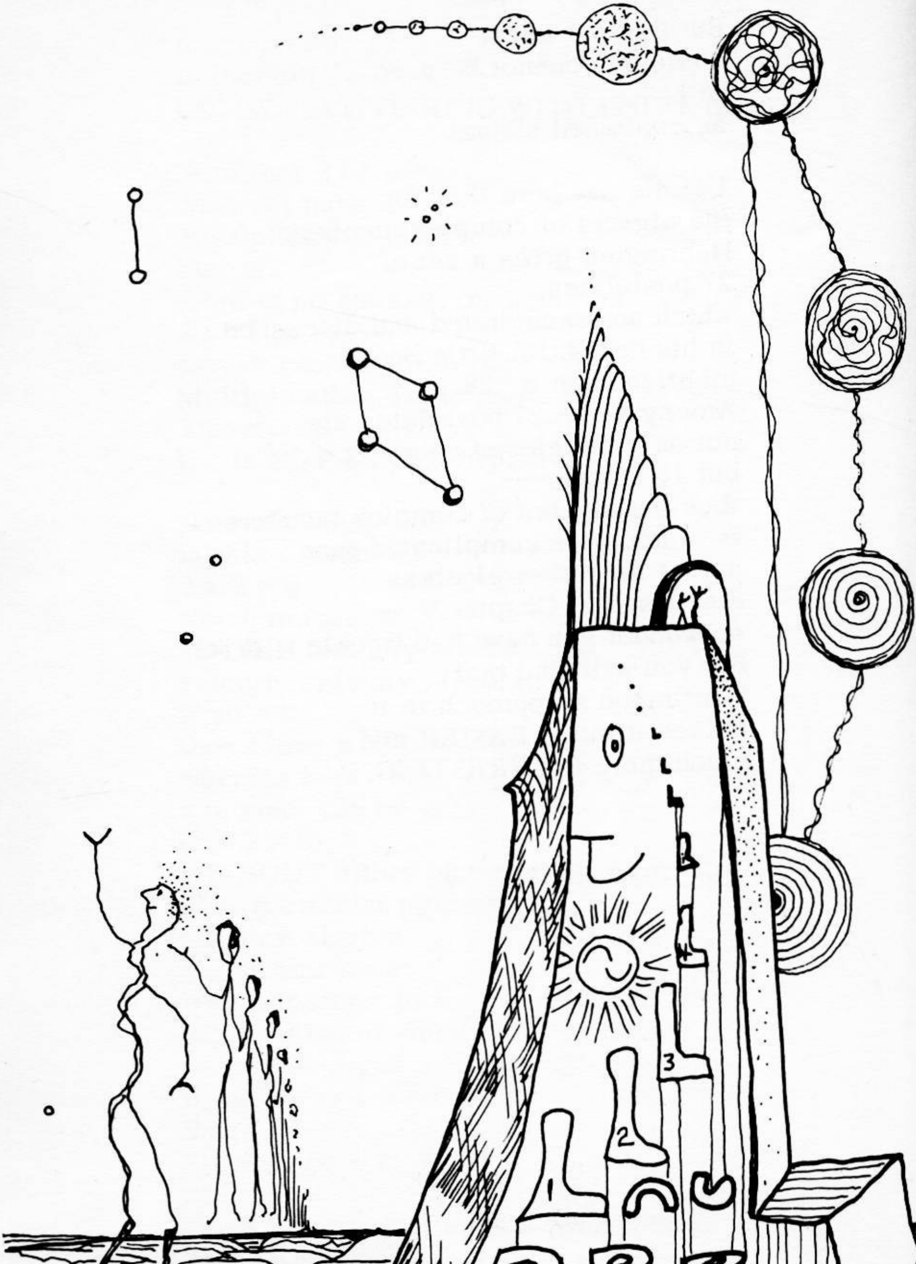
Note that 2 here
does not have a
multiplicative inverse,
that is,
there is no number, x, such that
$2x = 1$ (p. 49),
as you can see from the
Multiplication Table on p. 61.
Therefore
J_4 is NOT a FIELD (p. 54).

Note also that
in this system
$2 \times 2 = 0$
which makes it
UNLIKE ordinary algebra,
since in ordinary algebra,
if $ab = 0$,
then either a or b must = 0,
whereas here
a product can be zero
($2 \times 2 = 0$),
WITHOUT either factor being zero.
This particular property of
ordinary algebra
is, as you know,
very important in
the solution of equations,
where you argue as follows:
If $2x^2 - 5x + 2 = 0$,
then
$(2x - 1)(x - 2) = 0$,
and,
since this product equals zero,
THEREFORE
either $2x - 1 = 0$ or $x - 2 = 0$
which leads to the answers

$$x = 1/2 \quad \text{and} \quad x = 2.$$

But this argument
obviously cannot be used
in J_4 ,
as explained above!

Let me say here that for
the algebra of complex numbers
Huntington gives a set of
27 postulates,
which are enumerated and discussed
in his delightful little book,
mentioned on p. 28.
Among these 27 postulates are
not only the eleven on p. 53
but 16 more! —
thus the algebra of complex numbers
is much more complicated than
any of the newer algebras
discussed in Chapter V —
no wonder you have had trouble with it!
But you will find that
Huntington's approach to it
makes it much EASIER and
much more INTERESTING.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3

VII. A SUMMARY of PART I.

You have seen that
the Great Discovery at
the beginning of the
19th century ——
namely,
the realization that
the POSTULATES of a mathematical system
are NOT "self-evident truths"
but are
MAN-MADE ASSUMPTIONS,
and therefore
SUBJECT to CHANGE ——
had a profound effect on
mathematical creativeness,
for it gave to MAN
a NEW FREEDOM
which led to the development of
many NEW systems, ——
geometries, algebras,
and even logics,
which have enriched mathematics
beyond the wildest dreams and hopes
of previous centuries!

Furthermore,
I trust that the reader
already appreciates from Part I
(and he will do so increasingly
from the rest of the book):

1. The POWER of the
   POSTULATIONAL METHOD of THINKING,
   since an entire system is
   really contained in its POSTULATES ——
   from which all the
   THEOREMS or "consequences" follow!
   Contrast this with
   the so-called "thinking"
   in connection with many social issues,
   where no attempt is made to

clarify and make explicit
a few BASIC ideas,
from which many consequences flow!

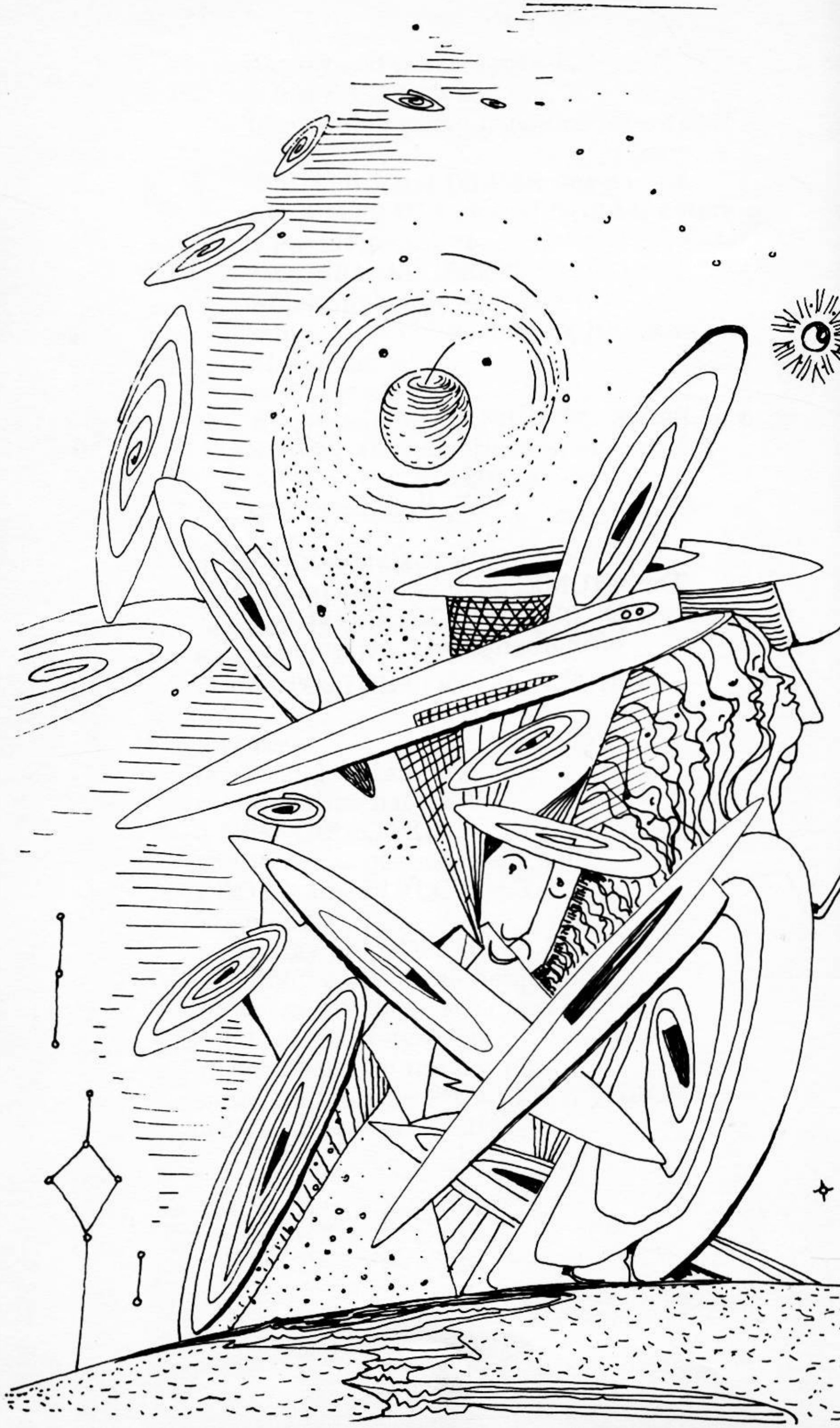
2. The additional POWER due to
the ABSTRACT and GENERAL nature of
a postulate set of
MODERN ALGEBRA,
which makes it possible to
make MANY different applications
of a given set —
see, for example,
the statement by Philip M. Morse on page 4,
as well as the definition of
ABSTRACT ALGEBRA
on page 145.

In short, consider the
"greater simplicity" and
"enormous economy in thinking",
as Garrett Birkhoff puts it,
in this POSTULATIONAL METHOD!

And now,
in the 20th century,
mathematics has taken
a NEW DIRECTION,
one which concerns itself with
ORDER and STRUCTURE
leading to
what may be called the
ATOMIC AGE of mathematics,
in which the inner structure of
mathematical systems
is examined by the newest of
powerful tools — LATTICE THEORY —
wait and see in Part II.

Part II.

The Twentieth Century:
The ATOMIC AGE of Mathematics:
ORDER and STRUCTURE
of mathematical systems.



VIII. Something NEW: ORDER.

Everyone knows,
no matter how far removed
he may be from science,
that in mathematics and science
there is a high degree of
ORDERLY thought,
deriving consequences from
basic postulates
in an orderly logical manner.
BUT it was not until
the twentieth century (!) that
the question of ORDER
in mathematics
was consciously examined and studied;
and this study has led to
some truly amazing results,
leading to the examination of
the detailed STRUCTURE
of various mathematical systems
so that we can actually visualize
"what makes them tick"!

As we shall see.

Of the 27 postulates of
the algebra of complex numbers
given by Huntington (see p. 64),
three concern themselves with
ORDER,
namely:

(1) If a and b are real numbers,
and if $a \neq b$, *
then EITHER $a < b$ OR $b < a$
(that is, if two numbers are NOT EQUAL,
then one of them MUST be less than
the other).

(2) If $a < b$, then $a \neq b$.

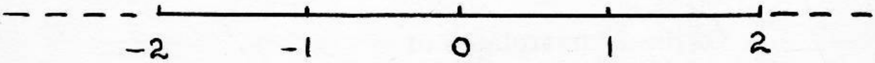
---

* The symbol $\neq$ means "is NOT equal to".

(3) If $a < b$ and $b < c$, then $a < c$.

Of course you will grant all these
without a struggle,
but you may be wondering
"Why the fuss?
Are they not
ridiculously simple, obvious and
unnecessary to mention?"
But wait till you see
situations in which they do NOT hold,
then the concept of
ORDER
will take on
a new meaning for you —
especially when you meet with
Lattice Theory,
where you will feel that
you have really crossed over into
the Twentieth Century!

Let us represent
the ordinary real numbers
by the points on a straight line,
in the usual manner:



Then,
for any two points, a and b,
$a < b$ shall mean that
a PRECEDES b or
a is to the LEFT of b;
and of course,
if two points are DISTINCT,
that is,
if we do not take the same point twice,

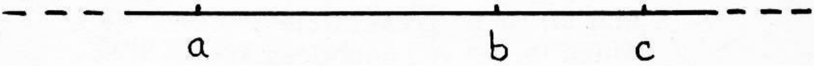
$$a \neq b,$$

then of course
either $a < b$ or $b < a$.
Thus postulate (1) on p. 68 holds here,

and so do (2) and (3),
as you can easily see.
When (3) holds, we may say that
b is "between" a and c,
and that
the three points a, b, c
are in the order

{abc} , *

thus



And you will no doubt grant that
"If points a, b, c are in the order {abc}
then they are NOT in the order {bca} ",
(i.e., c is here NOT between b and a),
which is one of
Veblen's postulates of ORDER
for ordinary Euclidean geometry.

You will be interested in
the following statement by Veblen
concerning the postulates of order
which he says
"were all used IMPLICITLY in
the older books on geometry
as well as in most textbooks of TODAY,
but have NOT been formulated
EXPLICITLY
as part of the foundations of geometry
until VERY RECENT times."

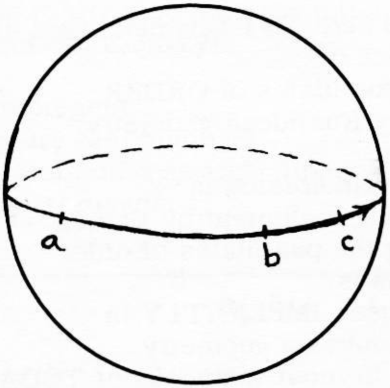
Again you may be wondering
why all the fuss about
something so simple and "obvious".
And again I say that

---

* See the splendid chapter
"The Foundations of Geometry"
by Oswald Veblen in
"Monographs on Modern Mathematics".

the easiest way to see its
IMPORTANCE
is to look at
a DIFFERENT geometry,
in which
this "obvious" postulate
does not hold at all!

Thus consider the NON-Euclidean geometry
on the surface of a sphere,
and consider three points a, b, c on
the "equator",
that is, on a "great circle",
which is, as you doubtless know,
a "geodesic"
(a shortest distance path)
on this surface. *



Here you can see that
b is between a and c,
that is,
the points are in the order {abc}
BUT, in spite of this,
is it not ALSO true here that
c is between b and a?! —
that these points are
ALSO in the order {bca} —

---

* See "Non-Euclidean Geometry" in this series.

since
if you start at b,
you may go to c, then to a,
continuing along the equator
in the direction shown!
And therefore
the "obvious" postulate on p. 70
does NOT hold in this
NON-Euclidean geometry
at all!

And so you see that
the postulates of ORDER
help to
DISTINGUISH one geometry from another,
and are therefore basic for
CHARACTERIZING a geometry.

Note also that the three points on
the surface of the sphere on p. 71
do NOT satisfy postulate (1) on p. 68
since BOTH $a < b$ AND $b < a$ hold here!

You will soon see that
this question of ORDER
is VITAL in the study of LATTICES
for which we shall need to know
what is meant by a
PARTIALLY ORDERED SET,
or POSET, for short
(as suggested by Garrett Birkhoff). *

But first
let me say a few words about
DIFFERENT KINDS of RELATIONS
and their properties;
this will help to clarify
everything that follows.

---

* See p. 1 of his book referred to
on p. 3 above.

(1) Take first the relation of
EQUALITY,
which is of course
very familiar to everyone.
It has, as you know,
the following three properties:

(a) $x = x$
(b) if $x = y$, then $y = x$
(c) if $x = y$ and $y = z$, then $x = z$

which may be briefly described by
saying that
EQUALITY is

(a) REFLEXIVE
(b) SYMMETRIC
(c) TRANSITIVE.

Not all relations have
these properties —
for instance,

(2) the relation $<$ (meaning "less than")
among real numbers:

(a) it is NOT reflexive
since it is NOT true that $x < x$;
(b) it is NOT symmetric,
since
if $x < y$, it does NOT follow that
$y < x$;
(c) but it IS transitive
since
if $x < y$ and $y < z$, then $x < z$.

(3) Again, take the relation

$$xRy$$

which shall mean
x "is the father of" y;
obviously this relation is:

(a) NOT reflexive,
that is,
x R x is FALSE since
x is NOT father of HIMSELF;

(b) NOT symmetric
since
if x R y is true, then
y R x is FALSE,
that is,
if x is the father of y, then
y is NOT the father of x;
and

(c) NOT transitive,
since
if x R y and y R z
(i.e., if x is the father of y
and y is the father of z)
then x R z is FALSE since
x is NOT "the father of" z
but his grandfather!
Here the relation R,
meaning "is the father of",
does not have ANY of
the three properties on p. 73.

Thus various relations between
the elements of a set
may have different properties,
and these are very important in a system.

Let us take one more relation
which has properties that are
very common in
mathematical systems,
namely:

(4) the relation $\leqq$ between
real numbers,
meaning
"is either less than or is equal to":
thus $2 \leqq 5$ (since 2 is $<$ 5),

and 3 $\overset{<}{=}$ 3 (since 3 = 3);
thus the symbol

$$\overset{<}{=}$$

allows EITHER < OR = to hold.
But 7 is NOT $\overset{<}{=}$ 5
since 7 is NEITHER < 5 NOR = 5.

This relationship $\overset{<}{=}$ has
the following properties:

(a) it is REFLEXIVE,
that is,

$$x \overset{<}{=} x,$$

is it not?

(b) if $x \overset{<}{=} y$ AND $y \overset{<}{=} x$,
then x must = y;
this property is called
ANTI-SYMMETRY. — only under the condition that $x \neq y$
Note that this property of
anti-symmetry
does NOT hold for
points on a great circle (p. 71).
BUT here we are dealing with
the ordinary real numbers —
extending to infinity in
both directions
as shown on p. 69 —
for which
if x is EITHER < OR = y,
and if, at the same time,
y is EITHER < OR = x,
then the ONLY possibility is
for x to EQUAL y.

(c) it is TRANSITIVE since

if $x \leqq y$ and $y \leqq z$, then $x \leqq z$. *

Now, a set which has
these three properties
with respect to some relation,
namely, if it is

P1: reflexive (e.g. $x \leqq x$);

P2: anti-symmetric
(e.g. if $x \leqq y$ and $y \leqq x$, then $x = y$);

P3: transitive
(e.g. if $x \leqq y$ and $y \leqq z$, then $x \leqq z$);

is known as a
PARTIALLY ORDERED SET
or POSET, for short. **

The significance of this term
will soon be evident.

More of this important concept of
PARTIALLY ORDERED SETS
in the next chapter,
where it will be illustrated by
VARIOUS relations
(other than $\leqq$) and

---

* For we have here the 4 possibilities:
(1) $x < y$ and $y < z$, then $x < z$ because
$<$ is transitive (by (2) (c) on p. 73)
(2) $x < y$ and $y = z$, then $x < z$
(by substitution)
(3) $x = y$ and $y < z$, then $x < z$
(by substitution)
(4) $x = y$ and $y = z$, then $x = z$ (p. 73)
And all 4 of these may be summarized in:
if $x \leqq y$ and $y \leqq z$, then $x \leqq z$.
Thus
the relation $\leqq$ is TRANSITIVE.

** Using P (as in P1, P2, P3) here
helps to remember that
these characteristics refer to a
Poset ——
as in G. Birkhoff's "Lattice Theory".

by some simple and interesting
DIAGRAMS,
often called Hasse diagrams,
which will then lead easily into
LATTICES.

IX. Partially Ordered SETS.

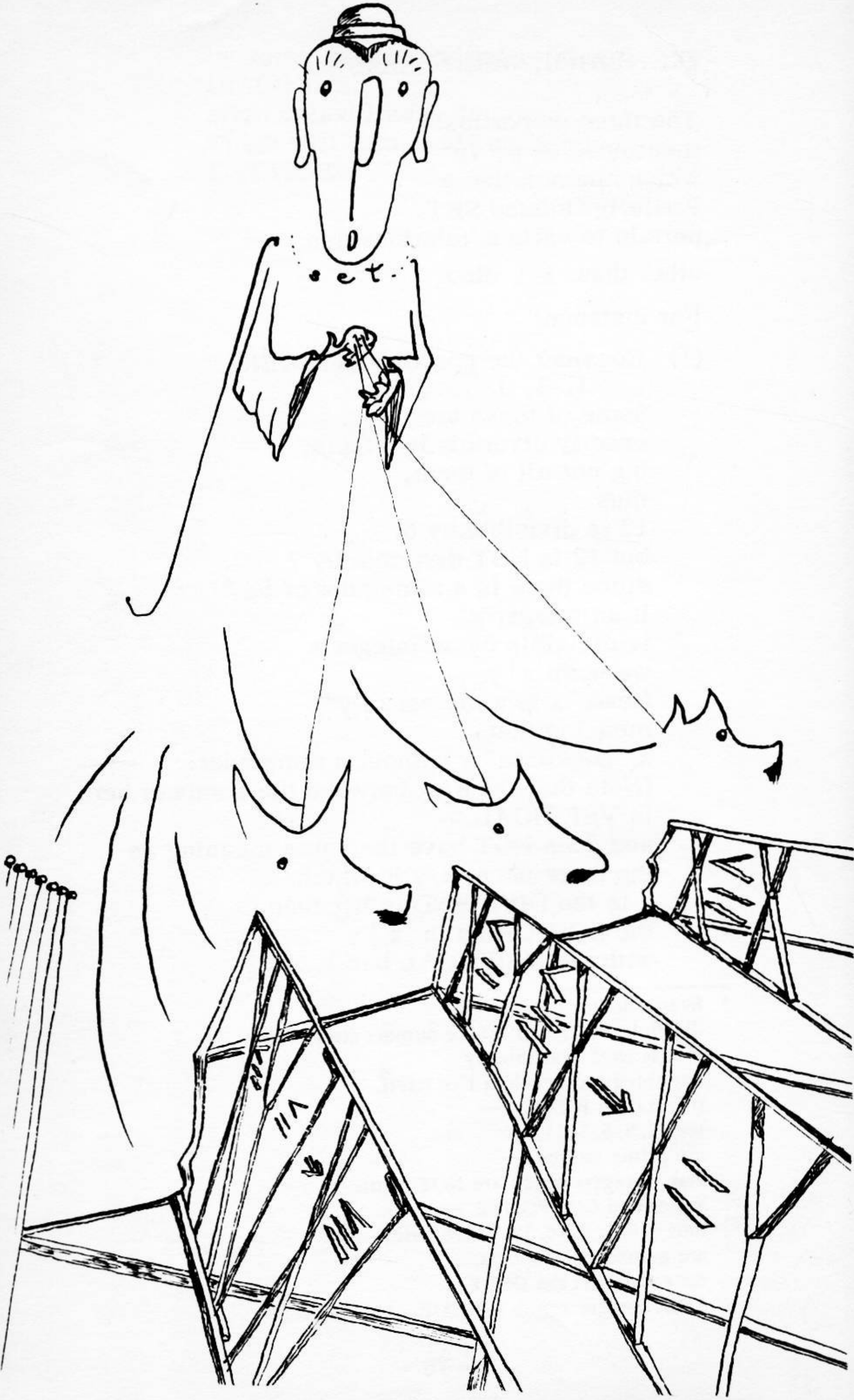
The three properties
mentioned on p. 76
which characterize a
Partially Ordered SET,
pertain to various relationships,
other than $\leqq$, also.

For instance:

(1) Consider the positive INTEGERS,
1, 2, 3, ...
Some of these are
exactly divisible by others,
but not all of them,
thus
12 is divisible by 6,
but 12 is NOT divisible by 7
since there is a remainder of 5. *
If an integer y
is divisible by an integer x
we write $x \mid y$
(read "x is a divisor of y")
meaning that
x "goes into" y without a remainder.
(Note that the BAR between the numbers here
is VERTICAL,
and does NOT have the same meaning as
the more usual x/y in which
x is the DIVIDEND rather than
the DIVISOR as in $x \mid y$
with the VERTICAL bar.)

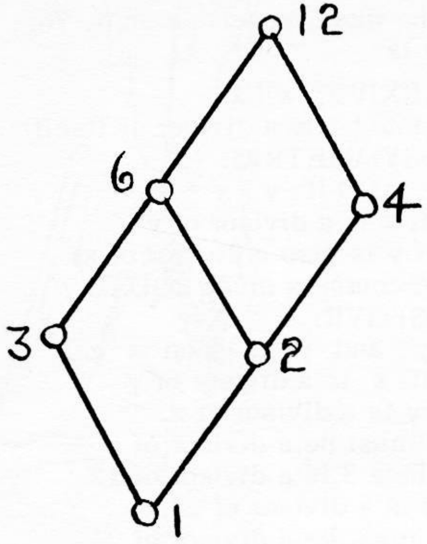
---

* Every integer is of course
divisible by 1, and by the number itself, —
if it is NOT divisible by
any integer other than 1 or itself,
it is called PRIME, —
thus 2, 3, 5, 7, 11, etc., etc.
are prime numbers;
and, integers which are NOT prime
are called COMPOSITE —
thus 4, 6, 8, 9, 10, 12, 14, 15, etc., etc.
are composite numbers.
Note that 2 is the ONLY
EVEN integer which is PRIME.


set

Observe that this relationship of
DIVISIBILITY
satisfies the three properties on p. 76,
namely, it is

P1: REFLEXIVE: $x \mid x$,
every integer is a divisor of itself;
P2: ANTI-SYMMETRIC:
if $x \mid y$ AND if $y \mid x$
(i.e. if x is a divisor of y
AND if y is also a divisor of x)
then of course x must EQUAL y;
P3: TRANSITIVE:
if $x \mid y$ and $y \mid z$, then $x \mid z$
(i.e. if x is a divisor of y
and if y is a divisor of z,
then x must be a divisor of z —
thus since 3 is a divisor of 12
and 12 is a divisor of 24,
then 3 must be a divisor of 24,
which of course it is).
Consequently the set of
positive integers
is a Partially Ordered SET
for the DIVISIBILITY relation.
Of course if you pick
two integers at random,
NEITHER one may be divisible by
the other,
like 5 and 7.
The following diagram shows clearly
the relationship of
Divisibility for the set of integers
consisting of 1, 2, 3, 4, 6, 12:



Here if a number x is
a divisor of a number y

$$x \mid y$$

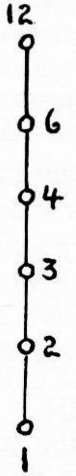
then x is BELOW y in the diagram
(unless of course $x = y$)
and is joined to it by
a RISING line —
either a line-segment or a broken line:
thus $2 \mid 4$, $2 \mid 12$, $2 \mid 6$, $3 \mid 6$, etc.,
but 4 is NOT a divisor of 6
and is therefore NOT joined to it by
a RISING line,
but by a line which goes
UP to 12 but then DOWN to 6.
And of course
3 is NOT joined to 4 by
a RISING line,
since 3 is NOT a divisor of 4.
You will find similar diagrams
(often called "Hasse diagrams"

"because of their effective use by H. Hasse")*
very useful in the study of
LATTICES,
as you will see.

(2) If these same numbers
(or any other set of real numbers)
are arranged according to the relation

$$\leqq$$

(instead of divisibility),
we saw on pp. 75ff that
this is also a POSET,
but the appropriate diagram
would then be:



where the numbers are arranged
one above the other,
in order of magnitude.

---

* See G. Birkhoff's "Lattice Theory", p. 6:
"The scheme goes back at least to H. Vogt,
'Résolution algébrique des équations',
Paris, 1895, p. 91,
and has been used for many years in
genealogy."

Note here that
of ANY two numbers,
one of them is SURE to be $\leqq$ the other;
whereas,
in the case of DIVISIBILITY,
you can pick a pair of numbers,
like 3 and 4,
NEITHER of which is
divisible by the other,
as shown in the diagram on p. 80.

Thus,
whereas BOTH illustrations,
on pp. 80 and 81,
are POSETS,
(i.e. Partially Ordered SETS) — p. 76,
still, as you see,
the relation $\leqq$,
UNLIKE divisibility,
satisfies not only
the three properties on p. 76,
but also the ADDITIONAL one:

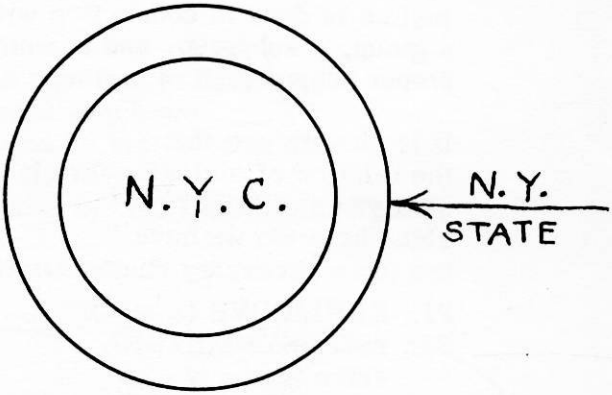
P4: For any two elements, x and y,
either $x \leqq y$ or $y \leqq x$

MUST hold.
Whenever this is the case,
the poset is said to be
"simply ordered",
and is called a
CHAIN,
like the one on p. 81 and
unlike the one on p. 80 which is
NOT a CHAIN,
though it IS a POSET.

(3) Let us next look at
another POSET
connected with the relation of
INCLUSION,
which is particularly important in
Boolean Algebra

(The Algebra of Classes)*
as well as in
Lattice Theory in general.

The inclusion relation is
a very simple concept concerning
classes —
a class is said to be
"included in" or "contained in"
another class if
every member of the first class is
also included in the second class:
thus
every person who lives in
New York City
also lives in
New York State,
and this fact may be represented
by the following diagram:



i.e. the class of
New York City dwellers
(call this class C)
is entirely within the class of
New York State dwellers

---

* Contrast Boolean Algebra with other Systems of Classes (pp. 179 and 180).

(call this class S);
and we express this by writing

$$C \subset S$$

(read "C is included in S").
Obviously any class is
entirely included within itself,
and therefore

$$C \subset C.$$

Thus, in general,
$x \subset y$ implies $x \leqq y$,
and x is said to be a
"sub-class" of y,
even when $x = y$.
If we wish to discuss
ONLY those sub-classes which are $< y$,
we speak of them as
"proper" sub-classes,
just as is done in connection with
a group, a subgroup, and a
proper subgroup. *

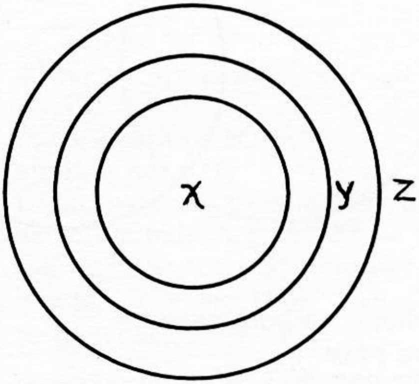
It is easy to see that
the relation of inclusion for classes
also gives a POSET,
since here too we have
the three necessary characteristics:

P1: REFLEXIVE ($x \subset x$);
P2: ANTI-SYMMETRIC,
    since if $x \subset y$
    (x entirely included in y)
    and if $y \subset x$
    (y also entirely included in x)
    then $x = y$,
    i.e. x and y must be the SAME class;
P3: TRANSITIVE,

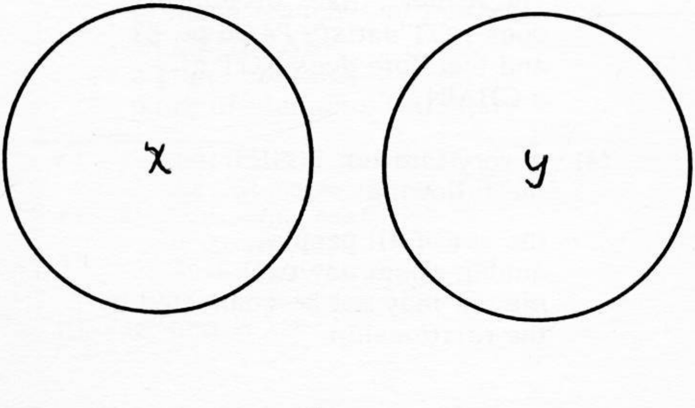
---

* See "Galois and the Theory of Groups"
in this series.

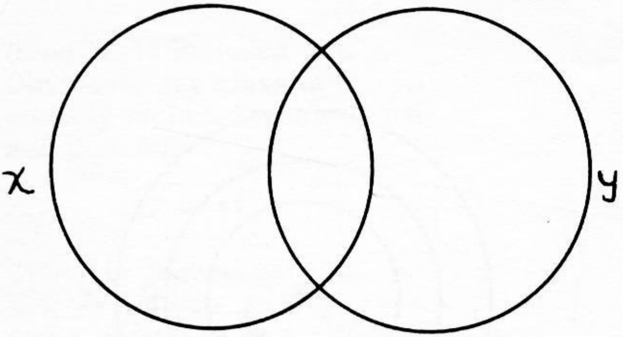
since if $x \subset y$ and $y \subset z$
then $x \subset z$,
i.e. x must be entirely included in z,
thus:



Furthermore
here it is possible to have
two classes which are
NOT related by inclusion at all,
INCOMPARABLE
(in the sense of "NOT COMPARABLE"),
thus:



or



where, in both cases,
we have
NEITHER $x \subset y$
NOR $y \subset x$,
i.e. x is NOT entirely within y,
NOR is y entirely within x —
just as we saw in the case of
Divisibility of integers (p. 80)
where we could find
two numbers, like 3 and 4,
NEITHER of which is divisible by
the other.

In other words,
"inclusion", like "divisibility"
does NOT satisfy P4 on p. 82
and therefore does NOT give
a CHAIN.

(4) A very familiar POSET is
the following:

the set of all people,
among whom any two
may or may not be connected by
the relationship

"is a successor (or descendant) of" —
provided that every person is
defined to be his own "successor". *
Note that this set
satisfies P1, P2, P3 on p. 76,
and is therefore a poset.

What is the use of all this?

Well —

A. So far as mathematics itself
is concerned,
the concept of POSETS
applies not only
to inclusion of classes (p. 82),
as in Boolean Algebra **
(as well as other Systems of Classes —
see pp. 179 and 180),
to the relation $\leqq$ of

real numbers (p. 74),
to divisibility of integers (p. 79),
but to many, many other systems
as, for example,

(a) the ideals of a ring, †
(b) the subgroups of a group, ††
(c) the subspaces of a vector space, †
etc., etc.

And so,
the concept of POSET
has proved to be valuable
in many branches of mathematics.
IN GENERAL,
a poset contains
pairs of elements which are

---

* See Hans Hermes: "Einfuhrung in die Verbandstheorie", p. 6.

** See "Mits, Wits, and Logic" in this series.

† See Birkhoff and MacLane: "Survey of Modern Algebra".

†† See Chapter XIX.

NOT COMPARABLE,
like the classes x and y on p. 85,
or the numbers 3 and 4 on p. 80;
in other words,
in general,
a poset is NOT a CHAIN (p. 82),
though there are many important ones
which ARE chains.
See the next chapter for
a great variety of diagrams
illustrating many kinds of
POSETS.
You will appreciate
their significance
in due course.

B. And, even outside mathematics,
as Garrett Birkhoff says: *
"The world around us
abounds with examples of
partly ordered sets.
Any hierarchy essentially forms
a partly ordered set."

The diagram on p. 80,
as well as those in the next chapter,
illustrate many different kinds of posets,
all of which,
in spite of their different appearances,
share in common
the properties on p. 76
which characterize a poset.

Thus,
all these different systems
can be studied by means of
this unifying concept
which sheds light upon them all,
as you will see.

---

* "Lattice Theory" by G. Birkhoff, p. 2.

Indeed,
you have seen similar diagrams
in the domain of genealogy,
in the organization of
a government or business,
which indicate the relationships
between the man at the top,
the president, etc.,
to the people in various departments
under him,
those immediately under him,
and those further down the line,
the "rank and file".

Thus you can see that
the study of the
STRUCTURE of an organization
can exhibit the parts and
their relationships with each other,
and thus improve
the effective use of the organization.

In Lattice Theory
this study of
ORDER and STRUCTURE
is done
not just superficially
as one does in a casual look at
a diagram,
but such amazing properties
are investigated,
which the casual observer
would not notice in
a million years!

And the realization of this,
whether you can use the results in
your daily job or not,
will give you a glimpse into what
THINKING
really means!

You know of course that
the symbol $\otimes$,
called "multiplication",
may represent any operation whatever,
remember? *
And even when it is written
without the circle around it,
like this, x,
it may still be used in
this more general sense.
Similarly,
the symbol $\leqq$
is used in connection with
POSETS
to represent VARIOUS relations,
not just "less than or equal to" —
but also
inclusion, divisibility, or
ANY other relation,
PROVIDED ONLY that
it satisfies the three basic
characteristics of a
POSET,
namely the properties of being

P1: reflexive
P2: anti-symmetric
P3: transitive

(see p. 76).

On pages 80 and 81
you saw two diagrams
representing posets,
one of which (p. 81) is a CHAIN,
and the other (p. 80) is NOT a CHAIN.

And you already know
that the difference between

---

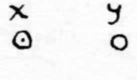
* See p. 27

these two types is that
for a CHAIN
ANY two elements MUST be
related by $\leqq$
(whatever this stands for),
whereas,
in a poset which is NOT a CHAIN,
there are elements which are
NOT so related,
NOT COMPARABLE
(as, for example, in the case of
Divisibility — see p. 82).

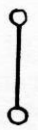
And this difference is
very clearly seen in
the diagrams themselves:
in a CHAIN
EVERY pair of elements is
connected by a RISING line (p. 81),
whereas
in a poset which is NOT a CHAIN,
there are SOME elements which
are NOT so connected
(like 3 and 4 on p. 80).

Let us now look at
some diagrams of posets:

(1)

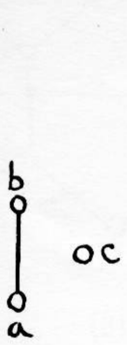


(2)

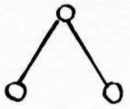


(3)

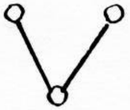
(4)



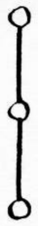
(5)



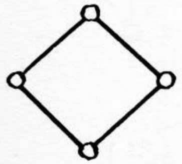
(6)



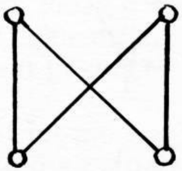
(7)



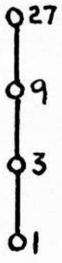
(8)



(9)



(10)



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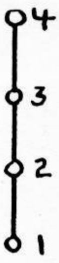

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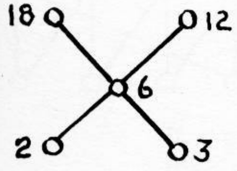

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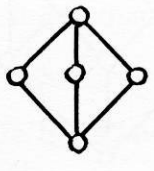

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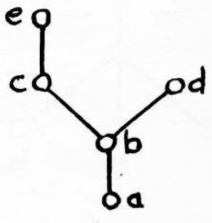

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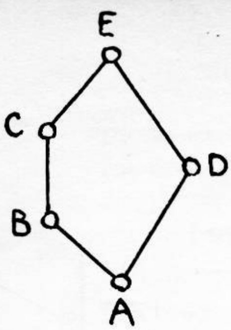

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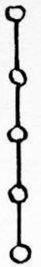

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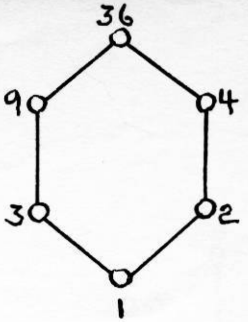

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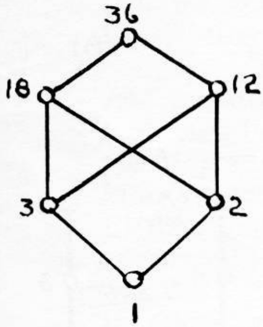

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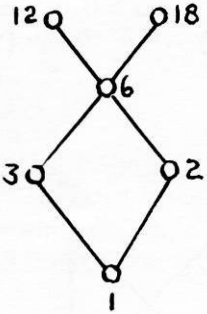

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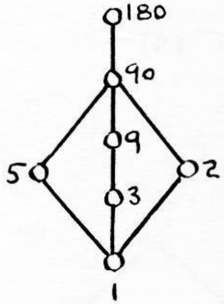
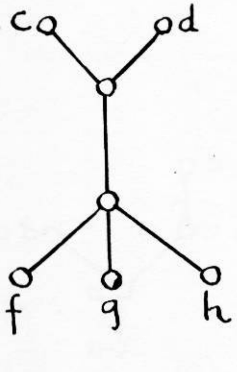

180
90
9
5
2
3
1

(21)



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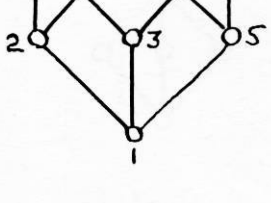
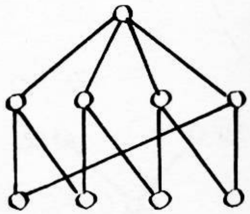

30
6
10
15
2
3
5
1

(23)



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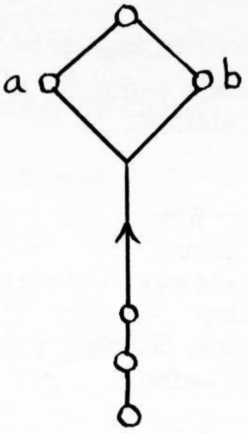

(25)

and many others are of course possible.
Now why do these diagrams,
in spite of their differences,
all represent posets?

First of all let me say that
in all these diagrams
the little circles represent
ELEMENTS
(whether they be numbers, or classes,
or any other);
and if one of these, say x,
is BELOW another, y,
in the plane of the paper,
and joined to it by
a RISING line,
then we shall say that

$$x < y.$$

If x and y are the same
then of course

$$x = y.$$

Thus, in general,

$$x \leqq y$$

shows that
EITHER x and y are
the SAME element
(if $x = y$)
OR
x is distinct from y and
is BELOW it and
is joined to it by
a RISING line
(in which case $x < y$,
as was said before).

But note carefully that
NEITHER of these relationships
between two elements, x and y,
must necessarily hold,
i.e. a pair of elements is
NOT NECESSARILY COMPARABLE:
thus in Fig. (2), p. 91,
NEITHER is $y \leqq x$
NOR is $x \leqq y$;
similarly,
for the elements 2 and 3 in
Fig. (19) on p. 93
(in this case it means that
of the numbers 2 and 3
NEITHER one is a DIVISOR of the other).
And there are many other such pairs
in some of the figures on
pages 91-94.
Note however that
there are NO such pairs in
(3), (8), (11), (12),
in which,
no matter what pair of elements
you select,
call them x and y,
you will find that
either $x \leqq y$ or $y \leqq x$;
hence these particular diagrams
represent CHAINS. (p. 82)

In ALL the diagrams on those pages
it is easy to see that
the following three characteristics hold:

(1) $x \leqq x$ (since $x = x$);

(2) IF $x \leqq y$ and IF also $y \leqq x$,
then obviously $x = y$;

(3) IF $x \leqq y$ and IF $y \leqq z$, then $x \leqq z$.

For example,
in Fig. (18) on p. 93,
since $3 < 9$
(i.e. 3 is connected to 9 by a RISING line)
and $9 < 36$
(9 is connected to 36 by a RISING line),
therefore
$3 < 36$
(i.e. 3 is connected to 36 by a RISING line),
meaning here of course that
since 3 is a divisor of 9
and 9 is a divisor of 36,
therefore 3 is a divisor of 36.
And similarly for
any other three numbers
related by $\leqq$

Thus every one of these
sets of elements
(represented by the diagrams
on pages 91-94)
has the characteristics of being

P1: reflexive
P2: anti-symmetric
P3: transitive

and is therefore a POSET.

In the next chapter
we shall take a closer look

at these diagrams,
and from them learn
some very interesting properties of
posets.
And later you will see also
how these diagrams are related to
LATTICES,
and how they can help in the study of
various mathematical systems —
geometries, groups, etc., etc.

XI. STRUCTURE of POSETS.

First of all
remember that when we say

$$x < y$$

this does not necessarily mean that
"x is less than y",
but only that
x is BELOW y on the page and
joined to it by a RISING line (p. 94),
and in fact may mean that
"x is a divisor of y",
as in (13), (18) and others on pp. 91–94.
Thus in (18), where
$\leqq$ refers to Divisibility,
we must NOT say $2 < 3$
(although 2 IS less than 3)
since 2 is NOT a DIVISOR of 3.
And similarly
when we say

$$x > y$$

we shall not necessarily mean that
"x is greater than y",
but only that
x is ABOVE y on the page and
joined to it by
a DESCENDING line,
and in fact may mean that
"x is divisible by y"
as in (18);
and therefore
we must NOT say here that

$$3 > 2$$

(even though 3 IS greater than 2)
since 3 is NOT DIVISIBLE by 2.

And of course these symbols
may have still other meanings
as stated on p. 90;
indeed in (15) on p. 92,
where NO specific meaning
(like "divisibility" or "inclusion", etc.)
is indicated,
we can still say that

$$a < b$$

and

$$e > c$$

and that c and d are
NOT COMPARABLE (p. 95) —
merely because of
their location on the page and
how they are joined to each other,
as explained on p. 98.
And, specifically,
if $z > y$ AND if
there is NO third element, x, such that
$z > x > y$,
then we say that
z "covers" y.
Thus in (15), p. 92,
d "covers" b,
because
there is NO element, x, such that

$$d > x > b;$$

but d does NOT "cover" a
even though $d > a$
because we have

$$d > b > a.$$

Thus we can study
the STRUCTURE of a POSET
even when it is ABSTRACT.

Let us now note
various characteristics of
the STRUCTURE of the diagrams on pp. 91-94:

(1) Some of the diagrams show
a "greatest" element —
as in (3), (6), (8), (9), etc.;
this means that
there is an element
(the "greatest")
which is > every other
element in the set,
in the sense explained on p. 98,
i.e.,
it is above them all on the page,
AND
is joined to each of them by
a descending line;
thus in diagram (5)
b is NOT the "greatest"
since it is NOT so joined to c.
Similarly in (13)
There is NO "greatest" —
the element 18 is NOT joined to 12 by
a descending line
(which means here that
18 is NOT DIVISIBLE by 12);
and in (15),
e is NOT the "greatest"
(even though it IS above
all the other elements)
BECAUSE
e is NOT joined to d by
a DESCENDING line.
Whereas (3), (6), (8), (9),
as well as (11), (12), (14), and others,
each DO have
a "greatest" element.
(Examine them and find
which other diagrams
DO have a "greatest" element
and which do not.)

(2) Similarly,
some of the diagrams on pp. 91-94
have a "least" element,

i.e.,
an element which is < every other
in the sense explained on p. 98.
Thus diagrams (3), (7), (8), (9),
and others,
DO have a "least",
whereas
(2), (4), (5), (6), and others
DO NOT.

(3) Note that
some of the diagrams have
BOTH a "greatest" and a "least",
as, for example,
(3), (8), (9), (11), etc.;
while some,
like (6) and (24)
have only a "greatest" but
not a "least";
and others,
like (7), (15), and (20)
have a "least" but NOT a "greatest".
And still others,
like (2), (4), (5), (10), (13), (22),
have NEITHER.

(4) Note further that
though some of the diagrams
do not have a "greatest" element,
they may nevertheless have
one or more "maximal" elements:
a "maximal" element is one which is
NOT < any other element
though it is
NOT NECESSARILY > than
all the others —

thus, for example
the element e in diagram (15)
is a "maximal" one since
it is NOT < any other,
and yet it is 92)
NOT > than d (p. 100).

Similarly,
element c in diagram (22)
is a "maximal" element
BUT NOT a "greatest" one
(it is NOT $>$ d);
also d in this same diagram
is a "maximal" element
though NOT a "greatest" one.

It is easy to see that
a "greatest" element is
always also a "maximal" one,
but NOT conversely,
i.e.,
a "maximal" one is
NOT NECESSARILY a "greatest" one,
as shown above.
And it is easy to see that
when there is a "greatest"
it is unique —
i.e. there CANNOT be more than ONE
"greatest" —
but of course there can be
more than one "maximal" element,
as in (22) on p. 93
which has NO "greatest",
but has two "maximal" elements,
c and d.

(5) There is a similar distinction between
a "least" element and a "minimal" one:
thus in (22) f, g, and h are
"minimal" elements
(for there is NO element $<$ than
f, g, or h),
but f, for instance, is NOT a "least" one
since it is NOT $<$ g —
and similarly for the other
"minimal" elements g and h,
neither of which is a "least" one.
But there CANNOT be MORE than
ONE "least", if any.

(6) It is sometimes convenient to
use the following terms:

a. ATOMS or POINTS are
the elements which "cover" (p. 99)
the LEAST, if there is one, *
like the elements 2, 3, 5 in (23) on p. 93;

b. HYPERATOMS ** "cover" ATOMS,
like 6, 10, 15 in (23);

c. ANTIATOMS are "covered" by
the GREATEST (if there is one),
like 6, 10, 15 in (23), etc.
(Note that these particular antiatoms
happen to be also hyperatoms,
though of course this is
not true in general.)

As you will agree,
all these are very simple properties of
these very simple diagrams.
And yet,
you will also doubtless agree, soon,
that these simple concepts will lead to
highly important and useful consequences!

But first let me give you
an interesting illustration of
a POSET which has
an INFINITE set of elements:
namely, the infinite set of
positive integers
arranged under DIVISIBILITY, thus:

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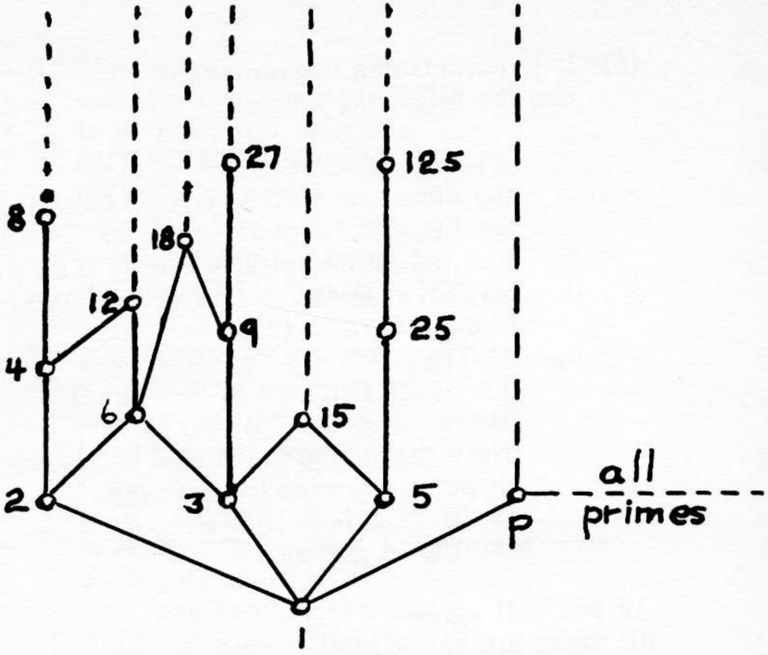
* See footnote on page 213.

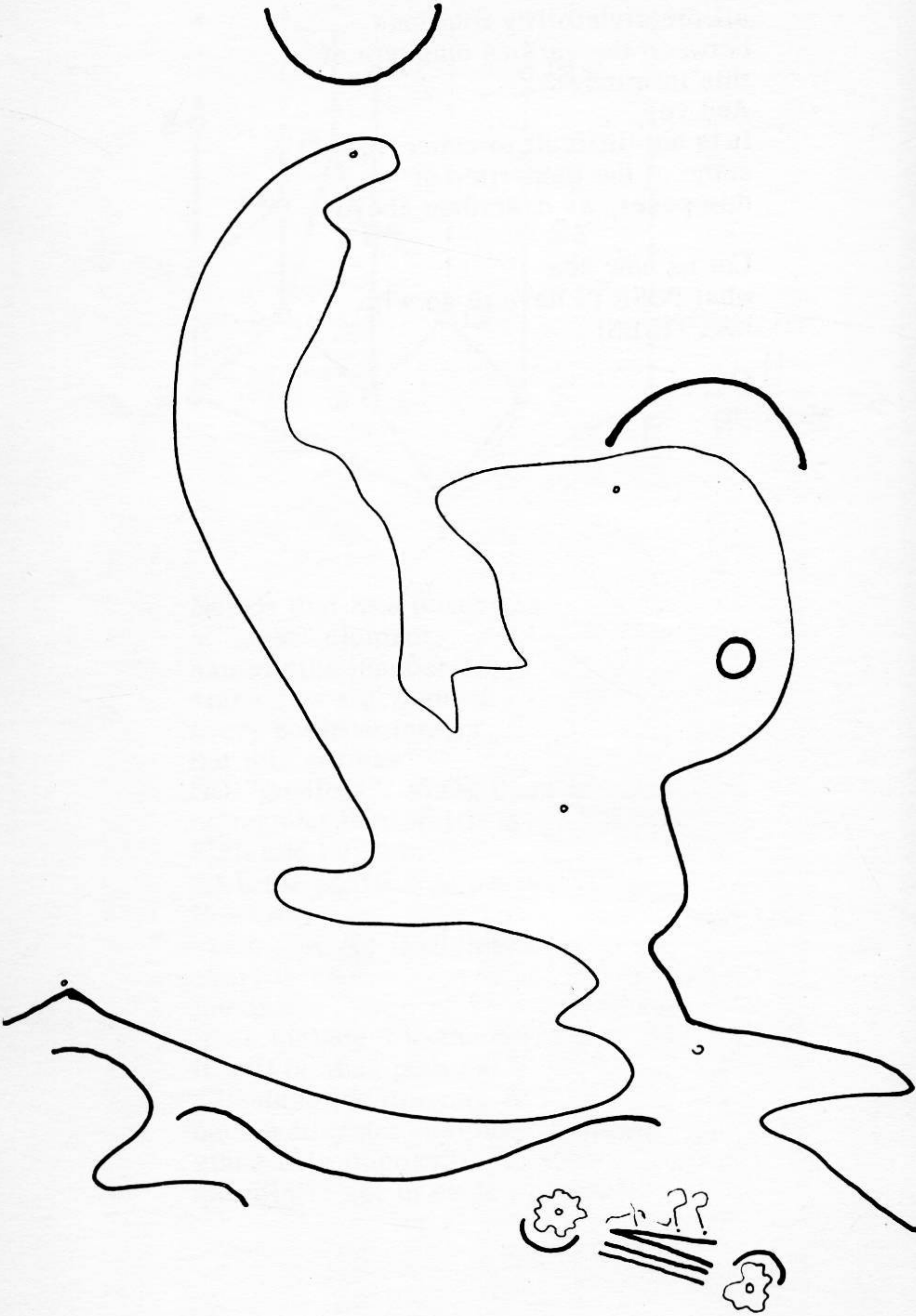
** A term used by H. Hermes in the book mentioned on page 87.



Notice that this poset has
a "least" element,
namely the number 1,
since 1 is a divisor of
every positive integer.
But this set has
NO "greatest", since there is
no number here which is
divisible by
ALL the positive integers.
However,
since zero (0) is divisible by
every integer,
therefore
if we include 0 in the set,
it will be the "greatest"!
Of course the diagram is
necessarily not completely shown
since it is impossible to show
an infinite set in such a diagram,

nor is it possible to show
all the divisibility relations
between the various numbers of
this infinite set.
And yet
it is not difficult to conceive
some of the properties of
this poset, as described above.

Let us now see
what POSETS have to do with
LATTICES!



XII. What is a LATTICE?

In the spirit of what was said
on p. 8, regarding
games and mathematical systems,
it is natural to ask

(1) what is the "equipment" of a lattice?
(2) what do we DO with it?
i.e. what are the operations here?
(3) what are the RULES of this "game"?
i.e. what are the
POSTULATES for a lattice?

Thus, abstractly,
a Lattice may be defined as *

(1) a set of elements a, b, c, . . . ,
(2) under two binary ** operations
represented by the symbols $\frown$ and $\smile$,
which are subject to
(3) the following three PAIRS of
POSTULATES:

L2 † : the commutative laws:
1. $a \frown b = b \frown a$ and
2. $a \smile b = b \smile a$;

---

* See Hans Hermes: "Einführung in die Verbandstheorie" (1955), p. 1.

** "Binary" refers to the fact that TWO elements are involved in an operation, as in $a \frown b$ or $a \smile b$.

† The reason for designating these postulates by the L's in this order is to make it conform to the nomenclature used in the 2nd definition of lattice, p. 107.
Using L's
of course reminds the reader
that these are postulates for
Lattices;
just as P's were used on p. 76.
to designate the characteristics of
Posets.

L3: the associative laws:
1. $(a \frown b) \frown c = a \frown (b \frown c)$ and
2. $(a \smile b) \smile c = a \smile (b \smile c)$;

L4: the "absorption" laws:
1. $a \frown (a \smile b) = a$ and
2. $a \smile (a \frown b) = a$.

Note that L2 and L3 are
analogous to the
commutative and associative laws
for multiplication and addition in
ordinary algebra,
but that L4 has no analogue in
ordinary algebra.
Note also that from L4
we can derive also

$$\text{L1:} \quad \begin{cases} a \frown a = a \text{ and} \\ a \smile a = a \end{cases} \quad *$$

known as the "idempotent" laws,
which also have NO analogue in
ordinary algebra,
where it is NOT true that

$$a \times a = a$$

NOR $a + a = a$.

An alternative definition of
a Lattice,
and one which shows
the relationship with
POSETS
(discussed in previous chapters)
is the following:

A Lattice is a POSET
any two of whose elements have

---

* By L4 no. 2, $a = a \smile (a \frown b)$;

$$\therefore a \frown a = a \frown [a \smile (a \frown b)]$$

$$\therefore a \frown a = a \text{ by L4, no. 1.}$$

Similarly $a \smile a = a$.

a "greatest lower bound" (g.l.b.),

$$x \frown y,$$

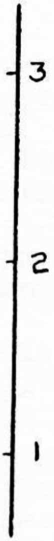
and a "least upper bound" (l.u.b.),

$$x \smile y. *$$

Let us first see
what these terms mean,
and then indicate how
this definition is equivalent to
the first definition, on p. 106.

In order to do this clearly
let us first recall
a few simple concepts
which you have already encountered
in other connections:
the concepts of
(a) "lower bound"
(b) "upper bound"
(c) "greatest lower bound"
(d) "least upper bound".

Thus, if you consider
the real numbers from 2 to 3 inclusive,



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* See "Lattice Theory" by G. Birkhoff (1948), p. 16.

you will agree that
2 is a boundary
below which they do not go,
and so
2 is a "lower bound" for
this set of real numbers, between 2 and 3.
But of course
they also do not go below 1,
so that 1 is also a "lower bound" for
this set of numbers.
In fact,
ALL numbers $\leqq$ 2 are
ALL "lower bounds" for
this set of real numbers.

Now
the GREATEST of all these
"lower bounds"
is obviously the number 2.
Thus 2 is the
"greatest lower bound" of
the set of real numbers from
2 to 3 inclusive.

Similarly,
this set of numbers has also
many "upper bounds"
namely
the number 3 and
all numbers greater than 3,
so that
3 is the "least upper bound" of
this set of real numbers.

Now what has all this
to do with
Lattices?

Well —
let us see first
how these concepts are
applied to POSETS.

Take first the
diagram (15) on p. 92
in which
consider the subset consisting of
the elements b and d:
here $b \leqq b$ and $b \leqq d$,

and therefore
b is a "lower bound" of
the subset consisting of
the two elements, b and d.
Also,
$a \leqq b$ and $a \leqq d$,
so that a is also a "lower bound" of
the subset b and d,
and there are no other
lower bounds of this subset.
Hence b is the
"greatest lower bound"
since $b > a$.
And similarly
c is the "least upper bound"
of the set b, c —
e being the only other "upper bound" of
this subset,
and $c < e$.

Now if we consider
the subset consisting of
c and d
it is easy to see that
b is the "greatest lower bound"
since b and a are both
lower bounds of the subset c, d
and $b > a$.
But this subset c, d
has NO "upper bound"
since there is NO element x such that
$x \geqq c$ AND $x \geqq d$
(d being a "maximal" element — see p. 101)
and hence of course
NO "LEAST" upper bound.

Whereas in diagram (16),
it is easy to see that
EVERY subset of TWO elements
has BOTH
a "greatest lower bound" (g.l.b.)
AND
a "least upper bound" (l.u.b.).
Thus, for the subset B, C,
the g.l.b. is B, and
the l.u.b. is C;
for the subset B, D
the g.l.b. is A,
and their l.u.b. is E,
and so on for any other
pair of elements.
Hence
the definitions of
$\frown$ and $\smile$ of two elements,
mentioned on p. 108 —
namely,
a $\frown$ b is the
GREATEST LOWER BOUND (g.l.b.)
and
a $\smile$ b is the
LEAST UPPER BOUND (l.u.b.)
of the two elements a and b.

In other words,

I. a $\frown$ b is defined by
the following two properties:

(1) $a \frown b \leqq a$ and $a \frown b \leqq b$;
which makes $a \frown b$
a lower bound of a and b;

(2) if $x \leqq a$ and $x \leqq b$,
i.e. if x is any other lower bound,
then $x \leqq a \frown b$ makes
$a \frown b$ the GREATEST lower bound (g.l.b.).

And similarly

II. $a \smile b$ is defined by:

(1) $a \smile b \geqq a$ and $a \smile b \geqq b$;
(2) if $x \geqq a$ and $x \geqq b$, then

$$x \geqq a \smile b$$

i.e. $a \smile b$ is the
LEAST upper bound (l.u.b.).

And so
IF EVERY pair of elements
of a poset
HAS a g.l.b., $\frown$, AND a l.u.b., $\smile$, *
then the system is a
LATTICE.

Let us now see
how this applies
to some concrete cases:

It is easy to see that
if the meaning of $\leqq$

happens to be
DIVISIBILITY,
then
$a \frown b$ (the g.l.b.) is the
GREATEST COMMON DIVISOR of a and b
in the given set,
and
$a \smile b$ (the l.u.b.) is their
LEAST COMMON MULTIPLE,
in the given set of course.
Thus in diagram (18) on p. 93

$3 \frown 2 = 1$ and $3 \smile 2 = 36$

(since 36 is the least multiple of
both 2 and 3
among the given set of numbers,
1, 2, 3, 4, 9, 36).

---

* $\frown$ and $\smile$ are sometimes called
"meet" and "join", respectively;
or also
"intersection" and "union",
as for classes.

Or, if $\leqq$ means

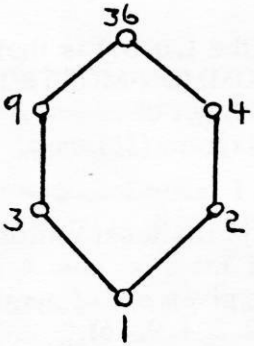
"less than or equal to",
as for real numbers arranged in
order of magnitude, *
then $a \frown b$ is
the smaller of the two, say, a,
and
$a \smile b$ is the larger one, b.
Thus, in diagram (12),

$2 \frown 3 = 2$ and $2 \smile 3 = 3$.

And similarly for other meanings of

$\leqq$ and $\frown$ and $\smile$.

Now, in (15), p. 92,
$c \frown d = b$,
but $c \smile d$ does NOT exist,
therefore (15) is
NOT a LATTICE.
Whereas in (18)
$3 \frown 2 = 1$ and $3 \smile 2 = 36$,
$2 \frown 9 = 1$ and $2 \smile 9 = 36$,
and so on for any pair of elements.
Therefore (18):



IS a LATTICE.

However, in Fig. (19):

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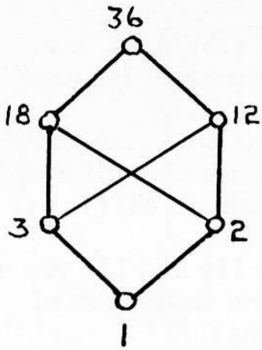
* See pp. 108 and 109.



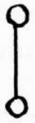
although the elements 12, 18, and 36
are all multiples of both 2 and 3,
and are therefore all
"upper bounds" of 2 and 3,
BUT
none of them is
a "LEAST upper bound", BECAUSE
though one might think that
12 is the "least" of the three
yet
it is NOT true that 12 < 18 here
since < here means
"is a divisor of" (p. 98).
Thus, of the "upper bounds",
12, 18, and 36
none is
a factor of BOTH the others —
in other words,
there is NO "LEAST" upper bound.
And consequently
Fig. (19) on p. 93
is NOT a LATTICE.

If you examine
the other diagrams on pp. 91–94,
you will see that
some of them
DO represent lattices,
like (3), (9), (16), etc.,
whereas others,

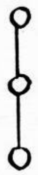
like (13), (15), (19), etc.
do NOT represent lattices.
Look carefully at
the various diagrams and
find those which ARE lattices and
those which are NOT.

On pages 116 and 117 you will find
a Complete Collection of
Lattices having
from 1 to 6 elements:

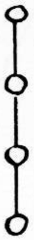
(1)



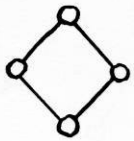
(2)



(3)



(4)



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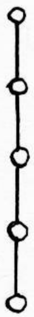
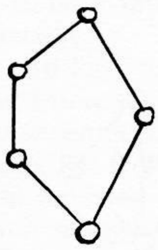


(6)



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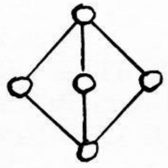

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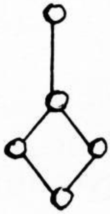

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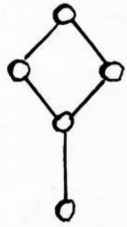

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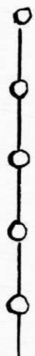

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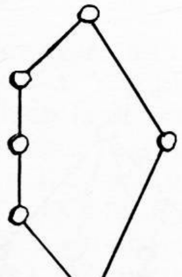
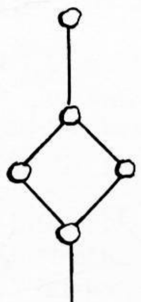
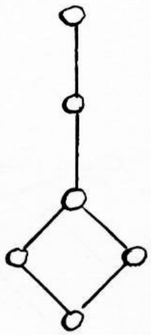


(12)



(13)



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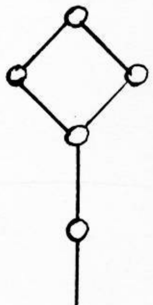
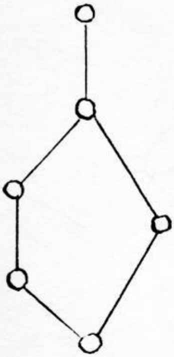
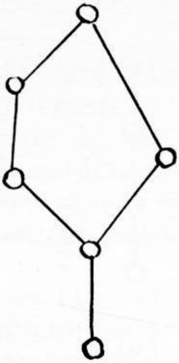


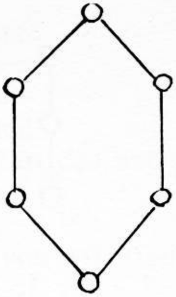
(15)



(16)



(17)



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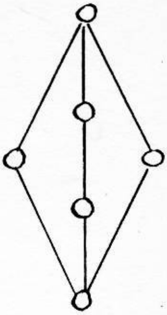

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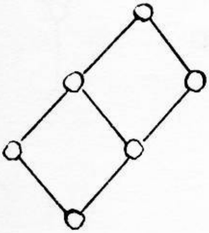
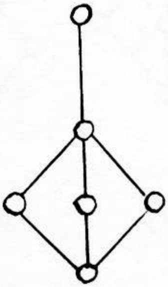


(20)



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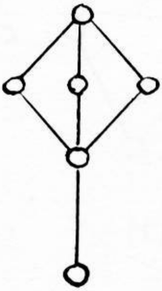

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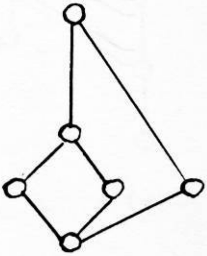

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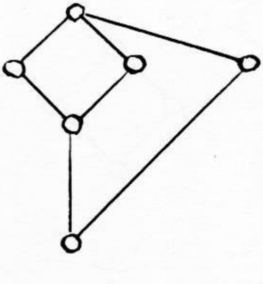
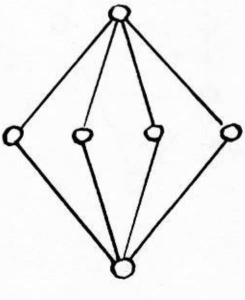


(24)



(25)

XIII. A Few Surprises!

It may seem that
in order to be a lattice
a poset must have
both a greatest and a least element,
since, for example,
diagram (15) CANNOT represent
a lattice
since c and d do NOT have a l.u.b. (p. 110),
as is required for
it to be a lattice (pp. 107, 108).
However,
if you look back at
the interesting INFINITE set
represented by the diagram on p. 104,
BEFORE 0 WAS ADDED TO THE SET,
you will see that
even though it has
NO "greatest" element,
still it IS a LATTICE,
because
every PAIR of integers HAS
a greatest common divisor (g.l.b.)
and a least common multiple (l.u.b.)
as demanded on pp. 107, 108. *

However, this lattice is
NOT a "complete" lattice,
which is defined as follows:
A poset in which EVERY subset has
a g.l.b. and a l.u.b.
is a COMPLETE lattice.
Thus the above illustration is
NOT a "complete" lattice,
since NOT EVERY SUBSET
(in this case the entire set)

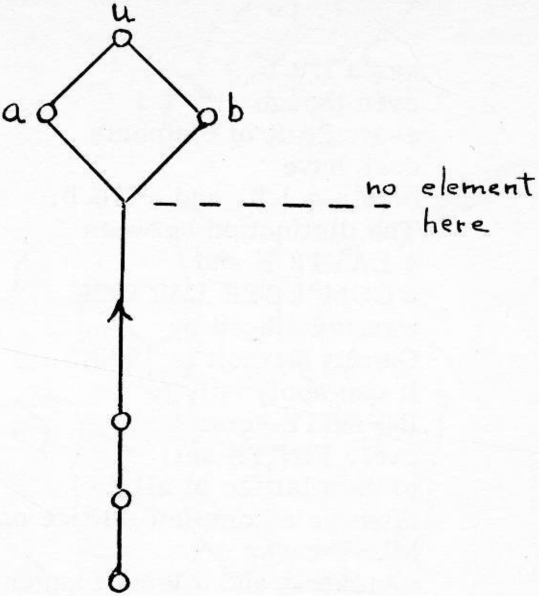
---

* Even two PRIME numbers,
like 3 and 7,
have the greatest common divisor, 1,
and a least common multiple,
in this case, 21.

has a l.u.b., $\cup$,
even though
every PAIR of elements
does have
both a g.l.b. and a l.u.b.
The distinction between
a LATTICE and
a COMPLETE LATTICE
was introduced by
Garrett Birkhoff in 1933.
It can apply only to
INFINITE sets:
every FINITE set,
to be a lattice at all,
must be a complete lattice and
MUST have
a greatest and a least element.
But the CONVERSE is NOT true,
i.e., a set,
whether FINITE or INFINITE,
which DOES have
BOTH a greatest and a least element
MAY still NOT be a Lattice.
Let me give two illustrations:

(1) One, the finite set in Fig. (19),
already discussed on p. 114.
And

(2) An infinite set,
represented by (25) on p. 94
and reproduced on page 120:



In this diagram,
the lower line with the arrow on it
represents the
INFINITE set of positive integers
arranged in order of magnitude,
which of course
has NO LAST element
(since there is NOT a greatest integer),
and consequently
the elements a and b have NO g.l.b. ($\frown$)!
So this too is NOT a LATTICE (see p. 107).
It is however a SEMI-lattice,
that is for $\smile$ ONLY! *
BUT
it is NOT a COMPLETE semi-lattice
since the subset represented by
the lower line with the arrow on it
has NO l.u.b. (see p. 118).

I mention these particular cases
because

---

* i.e. every pair of elements
DOES have a $\smile$,
though not necessarily a $\frown$

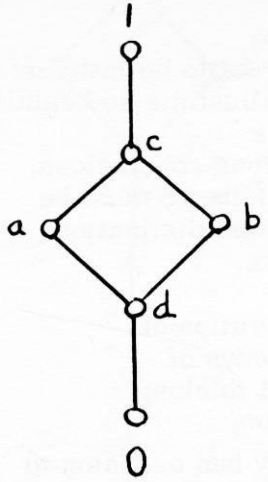
at first glance
they may appear to be lattices;
so that they illustrate so beautifully
how easy it is
to jump at wrong conclusions,
and how cautious we must be
to make delicate distinctions and
to avoid errors.

Another illustration of
the delicate ways of
mathematical thinking
is the following:

I have already had occasion to
refer to a "subset" of a set of elements,
meaning
some (or all) of
the elements of a given set.
And the reader may
already be familiar with
the concept of
a "subgroup" of a group,
meaning a subset of the elements of
the given group,
which themselves satisfy
the requirements for a group (pp. 48, 49).

And now
you may be surprised to know that
a subset of the elements of
a given lattice
may themselves satisfy
all the requirements for a lattice (pp. 106, 107),
and therefore actually BE a LATTICE,
and yet NOT be
a SUBlattice of the given lattice!*
Thus, consider the following lattice:

* Distinction between subset which is a lattice
and sublattice will be clear from example.



Here d and 0 are the only
"lower bounds" of a and b
and c and 1 are the only
"upper bounds" of a and b.
Further,
d is the "GREATEST lower bound"
of a and b,
(since $d > 0$), and
c is their "LEAST upper bound"
(since $c < 1$).
In other words,
$a \frown b = d$ and $a \smile b = c$.
But,
now take the subset of elements

$0, a, b, 1$.

Here 0 is the ONLY "lower bound"
of a and b,
and is therefore their
"greatest lower bound" (g.l.b.),
that is,
here $a \frown b = 0$
(not d, as before, since
d is not in this subset!).
And similarly, here

$$a \smile b = 1$$

(not c, as before).
And so,
though this subset is still
a Lattice
(since it still satisfies
all the requirements for a lattice — pp. 106, 107)
BUT,
since the ⌢ and ⌣ of a and b
do not give the same answers
as in the original lattice,
as explained above,
therefore this subset is
NOT a "SUBlattice" of the given one!

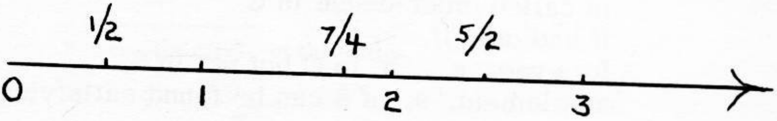
Another surprise in connection with
subsets:

Consider the set of positive integers,

1, 2, 3, ...

It is known as a "discrete" set,
meaning that
each element of this set,
has an "immediate successor" —
thus,
3 is the immediate successor of 2,
10 " " " " 9,
etc.

But,
if you consider the set of
positive rational * numbers,
arranged in order of magnitude



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* A "rational" number is one which is equal to the RATIO of two integers, like 2/3, 8/5, 4/1, etc., etc., — commonly known as "fractions".

it is easy to see that
this is NOT a "discrete" set,
since any one of these numbers,
say, 1/2, does NOT have
an "IMMEDIATE successor".
For if you choose any number here
which is greater than 1/2 (= .5),
say .6,
no matter how close to .5,
it is always possible to find
other rational numbers between them,
like .51 (= 51/100),
.52 (= 52/100) etc., etc.
In other words,
there is no rational number which is
RIGHT NEXT to
any given rational number.
Now,
such a set of numbers
is said to be
DENSE IN ITSELF.
Thus,
A CHAIN, C, is called dense-in-itself
if and only if,
given elements a and b of C such that

$$a < b,$$

there exists an element c in C satisfying

$$a < c < b.$$

Further,
A SUBSET, S, of a chain, C,
is called order-dense in C
if and only if,
for every $a < b$ in C but not in S,
an element, s, of S can be found satisfying

$$a < s < b.$$

Now the reader may be
surprised at the statement that

the set of all integers, J (p. 56),
has a countable * DENSE subset!
But it is easy to see that
the subset containing
the even integers only (call it E),
actually does satisfy the
definition of a dense subset given above —
thus
take any two numbers
in J but not in E,
namely,
any two ODD numbers,
then, between them
there is sure to be
some EVEN number,
is it not so?

And so,
though we started with
a DISCRETE set, J,
we find that it has
a DENSE subset!

This is another reminder that
we must not be too hasty in
jumping at conclusions —
for careful thinking has
MANY SURPRISES!

And now
let us look at the
basic characteristics of
LATTICES.

---

* See "Infinity", in this series.

X ≍ X = X
X ≬ (Y ≬ Z) = (X ≬ Y) ≬ Z
X ≬ Y = Y ≬ X
X ≬ (Y ≍ X) = X
≤
P
1, 2, 3,
X ≤ X
X ≤ Y ≤ X ⇒ X = Y
X ≤ Y ≤ Z ⇒ X ≤ Z

XIV. Basic Characteristics of LATTICES.

Having now seen
the meanings of the terms
g.l.b. and l.u.b. of
two elements, x and y,
or $x \frown y$ and $x \smile y$,
on pp. 111ff,
it is quite easy to verify
the following four pairs of
identities,
no matter what specific meanings
are given to $\frown$ and $\smile$ (pp. 112ff):

L1: $x \frown x = x$ and
$x \smile x = x$;

L2: $x \frown y = y \frown x$ and
$x \smile y = y \smile x$;

L3: $x \frown (y \frown z) = (x \frown y) \frown z$ and
$x \smile (y \smile z) = (x \smile y) \smile z$;

L4: $x \frown (x \smile y) = x$ and
$x \smile (x \frown y) = x$.

And you will recognize them to be
the "idempotent", "commutative",
"associative" and "absorption" laws
mentioned on pp. 106 and 107.

Thus the definition of
a LATTICE
on pp. 107ff
is equivalent to
the first definition given on p. 106
where the above-mentioned identities
were taken as postulates.

In either case,
these identities, L1 - L4,
completely characterize lattices,
and EVERY lattice satisfies them.

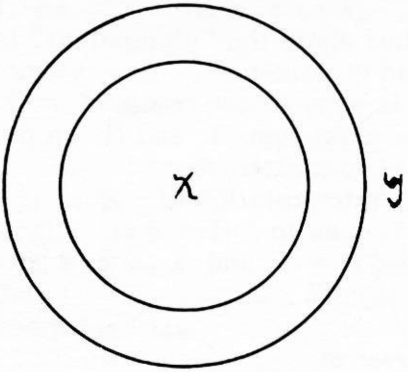
To make these identities seem
more concrete
let us again give to $\frown$ and $\smile$
some specific meanings:

(1) Take $\leqq$ to refer to "divisibility",
then $\frown$ gives the
greatest common divisor,
and obviously then
$x \frown x = x$
(i.e. the greatest common divisor of
3 and 3 is of course 3 —
and similarly for ANY number, x);
and also $x \smile x = x$,
i.e. the least common multiple of
5 and 5 is 5,
and of course also for any x.
Thus the "idempotent" laws, L1,
are valid here.

Again,
(2) Take $\leqq$ to mean
"less than or equal to",
then $\frown$ gives the lesser of
two given numbers, and
$\smile$ gives the greater one (p. 113),
so that
if you take two numbers
x and x (both alike),
then of course
$x \frown x = x$ and
$x \smile x = x$.
So here, too, L1 is valid.

And
(3) If we take $\leqq$ to mean
"inclusion" (p. 90),

thus



where x is entirely included in y,
then
for any PART of x,
call it z,
we have
$z < x$ and therefore $z < y$, **(p. 76)**,
so that z is a lower bound
of x and y **(p. 111)**,
and of course
the GREATEST "part" of x
is x itself,
so that
x is the GREATEST lower bound (g.l.b.)
of x and y,
i.e. $x \frown y = x$.
And consequently,
if y is equal to x,
then
x is the g.l.b. of x and x,
or $x \frown x = x$.
And similarly,
since
$x \smile y = y$ here,
therefore

if $y = x$,
we have
$$x \smile x = x.$$
Thus again the "idempotent" laws hold.
And of course
it is easy to see from
the definitions I. and II. on p. 111,
that no matter what
the interpretation of $\leqq$,
it is sure to follow that
$x \frown x = x$ and $x \smile x = x$,
is it not?

However,
these do NOT apply in
ordinary algebra, where
$x \cdot x \neq x$ and $x + x \neq x$,
since the operations of
x and + there
are NOT g.l.b.'s and l.u.b.'s,
as you very well know,
since they have nothing to do with
I. and II. on p. 111.

And now let us look at
L4, the "absorption" laws:
obviously they too do NOT hold in
ordinary algebra,
for + and x —
thus
$x \cdot (x + y) \neq x$ and
$x + (x \cdot y) \neq x$.
BUT
if we interpret $x \frown y$ to mean
the greatest common divisor of
x and y, and
$x \smile y$ their least common multiple (p. 112)
then $x \frown (x \smile y) = x$ DOES hold,
since $x \smile y$ is
the least common multiple (the l.c.m.)
of x and y,

and of course then
the greatest common divisor of
it (the l.c.m.) and x
is x itself —
thus
if $x = 2$ and $y = 3$
then $2 \smile 3 = 6$ and
$2 \frown 6 = 2$,
i.e.
$2 \frown (2 \smile 3) = 2$,
or, in general,
$x \frown (x \smile y) = x$
which is the
FIRST "absorption" law.

And similarly for
the SECOND "absorption" law,
e.g. $2 \frown 3 = 1$ and
$2 \smile 1 = 2$,
hence $2 \smile (2 \frown 3) = 2$,
or, in general,
$x \smile (x \frown y) = x$, which is the
SECOND "absorption" law.

And of course
the same would follow if
$\frown$ and $\smile$ were given other
possible interpretations
as on p. 127.
Try them and see.

As regards L2 and L3,
the commutative and associative laws,
these are obvious from
the definitions of $\frown$ and $\smile$ on pp. 111, 112
(as well as from
the various special meanings of
$\frown$ and $\smile$ above).
And, incidentally,
these commutative and associative laws
hold also for
ordinary + and x in
ordinary algebra.

But in spite of this,
ordinary algebra is of course
NOT a lattice, since
it does NOT satisfy
ALL of L1 - L4
(namely, it does not satisfy
L1 and L4, as shown on p. 129).

Now, if you compare a
LATTICE (p. 106)
with some of the algebras
described in Chapter V.,
you will notice that
a LATTICE has
relatively few postulates
(the algebra of complex numbers has
27 postulates,
remember? — see p. 64).
And yet,
like a group
(which has only
FOUR postulates — see pp. 48ff*),
it has a great many
important applications,
and is fast becoming
one of the basic modern tools of
20th century mathematics.

Perhaps by the time you finish
this little book,
you will have some idea of
how Lattice Theory
helps to exhibit the
STRUCTURE of
various branches of mathematics.
One of the important tools in
making this possible
is the concept of MAPPING.
So, before going into any applications,
let us see what is meant by
MAPPING, in mathematics.

---

* See also Appendix I., page 267.

XV. MAPPING.

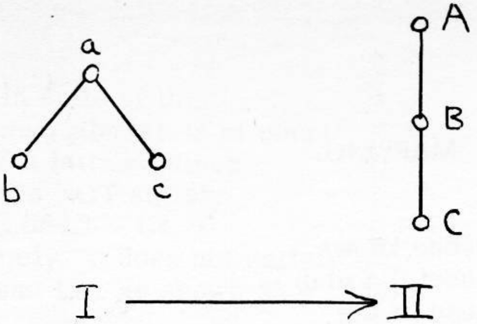
Everyone knows
how useful a map is,
because of the
one-to-one correspondence
between
the actual places of a territory
and the corresponding points
on the map,
so that
by following the map
you can guide yourself
through the country!

In mathematics
this useful concept
has been further developed
so that it is even more useful!

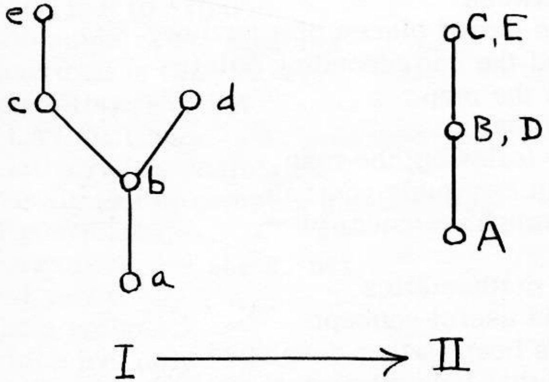
Before "selling" you
this new usefulness,
let us "play around" with
the idea of MAPPING —
just as a musician
"plays around" with a tune,
before it becomes so "useful" for
selling tooth-paste!

So let us consider
the following mapping of
POSETS in which
an element x in I. is
"mapped" onto the corresponding
element X in II. :

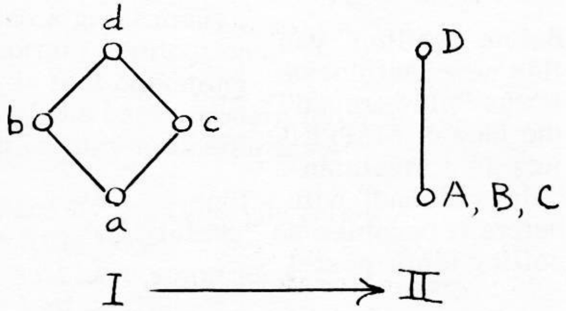
(1)



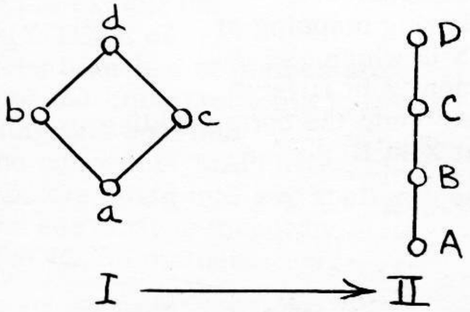
(2)


e
c
d
b
a
C, E
B, D
A
I
II

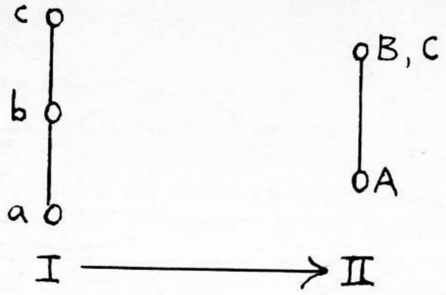
(3)


d
b
c
a
D
A, B, C
I
II

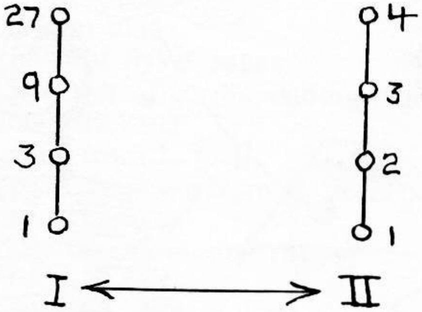
(4)


d
b
c
a
D
C
B
A
I
II

(5)

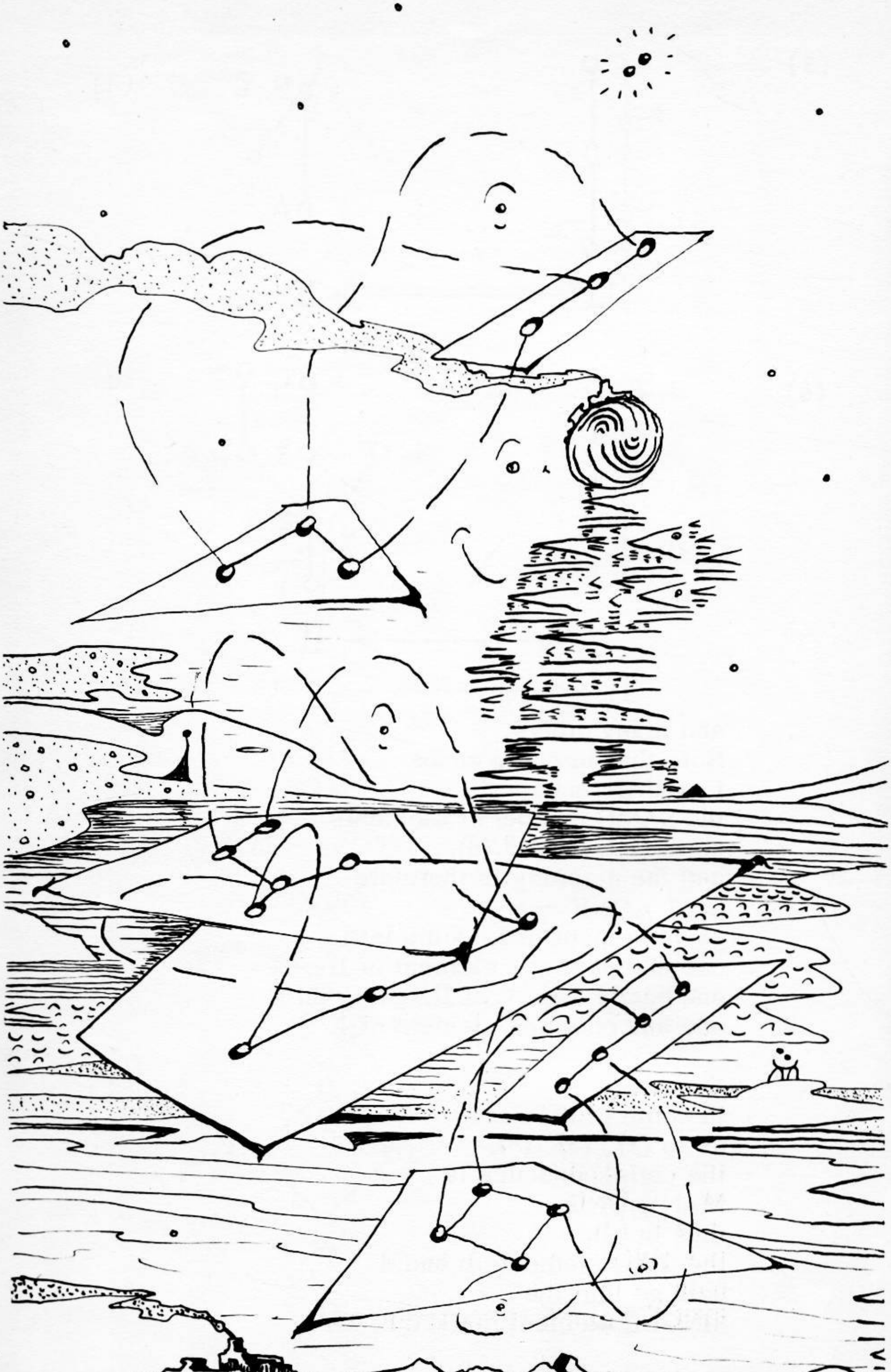


(6)



and many others.
Note that in some cases
I. and II. have
the SAME number of elements,
as in (1), (4) and (6),
and the mapping is therefore
ONE - ONE —
each element of I. going into
one and only one element of II.,
and each element of II. going into
one and only one element of I.

Whereas
in some of the cases,
as in (2), (3), (5),
the correspondence is
MANY-ONE:
thus in (2),
the TWO elements b and d
both go into the
SINGLE element marked B, D;



c and e both go into
the SINGLE element marked C, E;
in case (3),
a, b, and c all three go into
the SINGLE element marked A, B, C;
and in (5),
b and c both go into
the SINGLE element B, C.

Note also that
in the first FIVE cases
the ORDER of corresponding elements
is PRESERVED
in going from I. to II.
but NOT in going from II. to I.:
thus in case (1),
for any two elements which
are ordered in I.,
like $b \leqq a$ and $c \leqq a$,

we have the corresponding

$$B \leqq A \quad \text{and} \quad C \leqq A,$$

BUT
though $C \leqq B$ in II.,
we do NOT have $c \leqq b$ in I.;
similarly in (2),

$$b \leqq d \longrightarrow B \leqq D$$

(even though b and d are
distinct elements,
while B and D are the same here,
both situations are
taken care of by the symbol $\leqq$,
are they not?),
while

$D \leqq C$ in II. does NOT give $d \leqq c$ in I.

And so on for cases (3), (4), (5).

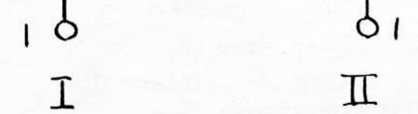
Whereas, in case (6),
the ordering is maintained in
BOTH directions, —

of any two elements x and y of I.
for which $x \leqq y$,
we have also $X \leqq Y$ in II.,
and conversely,
for any $X \leqq Y$ in II. we have also
$x \leqq y$ in I.

Now whenever
the ORDER is preserved in
ONE direction ONLY,
the mapping is called a
HOMOMORPHISM,
regardless of whether it is
one-one (as in case (1)) or
many-one (as in case (2)).

And
if ORDER is preserved in
BOTH directions
AND IF
the correspondence is
ONE - ONE, as in case (6),
the mapping is called an
ISOMORPHISM.

And of course this would apply
even if $\leqq$ refers to, say,
"divisibility" in the one poset and
"less than or equal to" in the other poset,
as in (6) on p. 134:



set up the 1-1 correspondence:

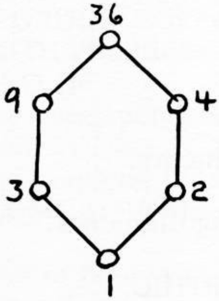
$$27 \longleftrightarrow 4$$
$$9 \longleftrightarrow 3$$
$$3 \longleftrightarrow 2$$
$$1 \longleftrightarrow 1$$

and interpret $\leqq$ to mean
"divisibility" in I. and
"less than or equal to" in II.;
then $3 < 9$ in I. means of course
3 "is a factor of" 9,
and
$2 < 3$ in II. means
2 "is less than" 3,
and yet
both MAINTAIN
the correspondence of elements,
and similarly for
ANY pairs of corresponding elements.
And so
these POSETS are ISOMORPHIC,
as indeed is easily seen
from the fact that
their diagrams are identical!
The converse, however, is
not necessarily true,
that is,
two posets may be
isomorphic
WITHOUT having identical diagrams
(see p. 215).

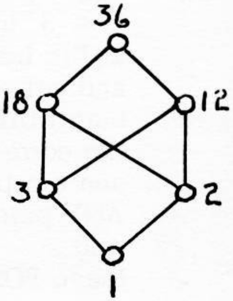
Thus obviously (9) and (10) on p. 92
are NOT isomorphic
since,
even though they each have

four elements,
and therefore
a 1-1 correspondence
can be set up
(in various ways),
still
no matter how this correspondence
is set up,
the ORDER of the elements
CANNOT be preserved —
which is of course obvious
from the diagrams.

Similarly for the diagrams (p. 92)
(13), (14), (15), (16) and (17),
although they each have
five elements;
and for (21) and (22), etc.
Compare also the posets (18) and (19):



(18)



(19)

and set up the 1-1 correspondence:

36 ⟷ 36
9 ⟷ 18
4 ⟷ 12
3 ⟷ 3
2 ⟷ 2
1 ⟷ 1

and $\leqq$ means
"divisibility" in both these posets.
And yet
whereas in (19)
$3 < 12$
(i.e. 3 is a factor of 12
and is therefore joined to it by
a RISING line),
still their corresponding elements,
3 and 4 in (18)
are NOT so related,
i.e. 3 is NOT a factor of 4.

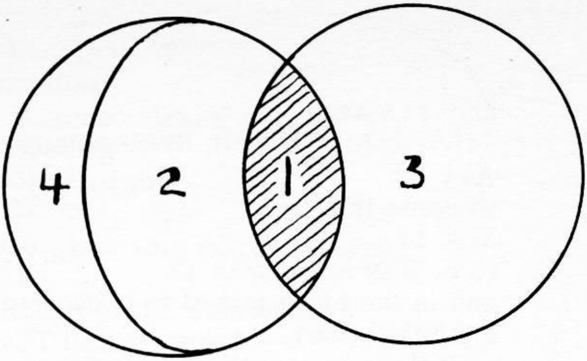
Similarly,
in (19), $2 < 18$, whereas
the corresponding elements in (18),
namely, 2 and 9,
are NOT so related,
i.e. 2 is NOT a factor of 9.

In other words,
in spite of the 1-1 correspondence,
these posets are
NOT ISOMORPHIC since
the relationship $\leqq$ is
NOT PRESERVED.
And of course this is
reflected in the diagrams
because the two additional lines in (19)
make this diagram
non-isomorphic to (18)
as explained above.

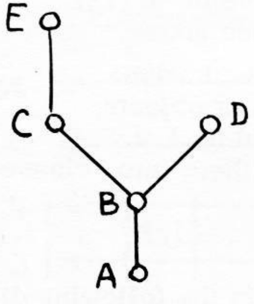
Another illustration:
consider four objects,
represented by 1, 2, 3, 4;
let us put them into "classes", thus:

$$\left\{1\right\}, \left\{1, 2\right\}, \left\{1, 3\right\}, \left\{1, 2, 4\right\},$$

as shown in the following diagram:



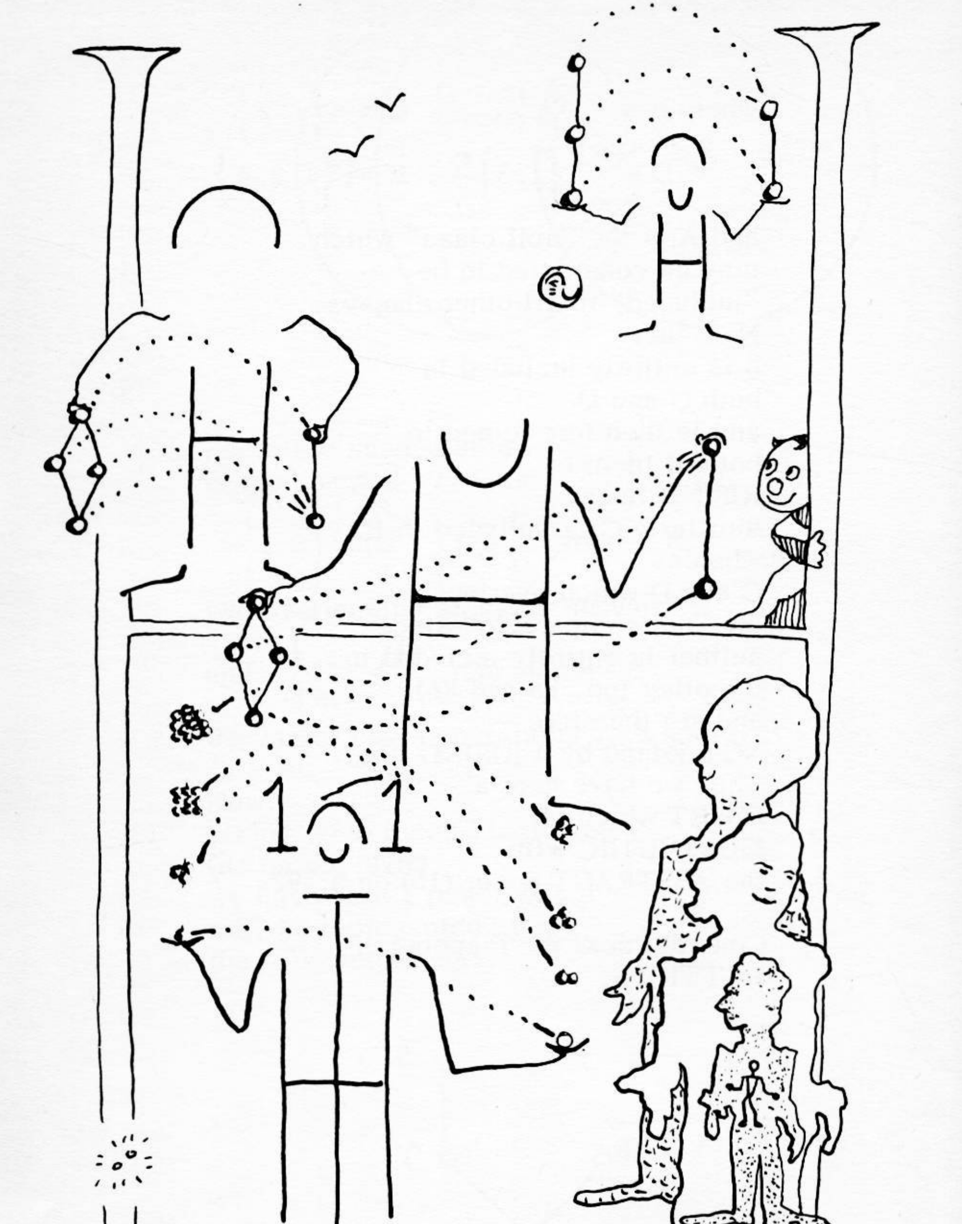
where the shaded area
is represented by class $\left\{1\right\}$;
$\left\{1, 2\right\}$ and $\left\{1, 3\right\}$ represent
two overlapping areas as shown,
and $\left\{1, 2, 4\right\}$ represents
the larger class containing 1, 2 and 4. To this add the null set also.
Now,
taking $\leqq$ to mean "inclusion"
(as on pp. 82ff)
we have here a poset which
may be represented by
the following diagram:



where B = $\{1\}$, C = $\{1, 2\}$
D = $\{1, 3\}$, E = $\{1, 2, 4\}$

and A is the "null class" which
may be considered to be
"included" in all other classes.
Note that
B is entirely included in
both C and D,
and is therefore joined to
both of them by
RISING lines;
similarly C is included in E;
whereas
C and D which overlap
are "not comparable" since
neither is entirely included in
the other (pp. 85 and 86),
and are therefore
NOT joined by a RISING line.
Thus we have here a
POSET which is
ISOMORPHIC with
the ABSTRACT poset (15) on p. 92.

Consider next the mapping of
LATTICES.

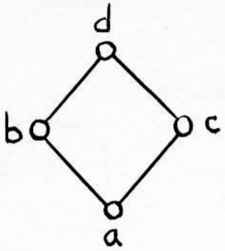

101

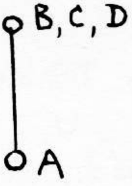
XVI. The Mapping of LATTICES.

In the mapping of lattices
we must examine also
the results of
performing the operations $\frown$ and $\smile$,
to see whether
the correspondence is maintained.
Thus, in case (3), p. 133,
note the following:
for any pair of elements x, y in I.,
such that $x \frown y = z$,
then correspondingly in II.
$X \frown Y$ gives the element which
corresponds to z, namely Z.
Thus
$a \frown b = a$ and $A \frown B = A$,
$b \frown c = a$ and $B \frown C = A$,
and similarly for any other
pair of elements.

However,
note that this correspondence
does NOT hold in case (3) for
the operation $\smile$:
for instance,
whereas $b \smile c = d$
still $B \smile C \neq D$.
Such a mapping is called
a $\frown$ -homomorphism
(also called a
"meet"-homomorphism — see p. 112),
but it is NOT a $\smile$ – homomorphism
(or a "join"–homomorphism).

Similarly it is possible to have
a $\smile$ –homomorphism which is
NOT a $\frown$ –homomorphism,
as in:





Test it and see.

It is possible also to have
a $\frown$, $\smile$ –homomorphism
(sometimes called a
"lattice"–homomorphism)
as in case (5) — p. 134,
where
$a \frown b = a$ and $A \frown B = A$,
$a \frown c = a$ and $A \frown C = A$,
$b \frown c = b$ and $B \frown C = B$;

and also
$a \smile b = b$ and $A \smile B = B$,
$a \smile c = c$ and $A \smile C = C$,
$b \smile c = c$ and $B \smile C = C$,

so that this is a
$\frown$, $\smile$ – homomorphism
as stated above.

And finally,
it is possible to have
a mapping of
one lattice on another
which is
NEITHER a $\frown$ – homomorphism
NOR a $\smile$ – homomorphism,
as in case (4) on p. 133
where
$b \frown c = a$ whereas $B \frown C = B$ (not A)
and
$b \smile c = d$ whereas $B \smile C = C$ (not D)

The idea of isomorphism
is applicable
not only to posets and lattices
but also to any other systems —
the advantage being that
when you study a system
in the ABSTRACT,
that is,
study its STRUCTURE,
it can then be applied to
MANY different situations
which are ISOMORPHIC with it * —
depending upon your needs.

Thus
the four postulates for a group
(see p. 48)
are of course abstract,
that is,
they do not say
what the elements a, b, c, ...
stand for,
they do not say
what the operation, o, is, —
and yet the abstract theorems
are derived from these
by means of logic
without giving specific meanings to
a, b, c, ... , and the operation, o.
But THEN
we can give specific
concrete meanings
to the elements and the operation **
and thus apply

---

* See p. 4.

** As Galois did when he used
SUBSTITUTIONS for ELEMENTS and
the operation was
FOLLOWING one substitution by another.
And this was the remarkable tool
with which he solved
so many important problems
(see p. 52).

this abstract system to
all kinds of problems which are
isomorphic with the abstract one.

So we see that
this trick of
ISOMORPHISM
makes an abstract tool usable
by many people
for many purposes!
Is not then the ABSTRACT
the most PRACTICAL
of tools?

Thus
ABSTRACT ALGEBRA
may be defined as
the study of those properties of
algebraic systems which
are preserved under
ISOMORPHISM. *

Furthermore,
HOMOMORPHISM, described above,
is another powerful tool in
the study of
various branches of mathematics.
It leads, via many-one mappings
to the fascinating concept of
IDEALS —

---

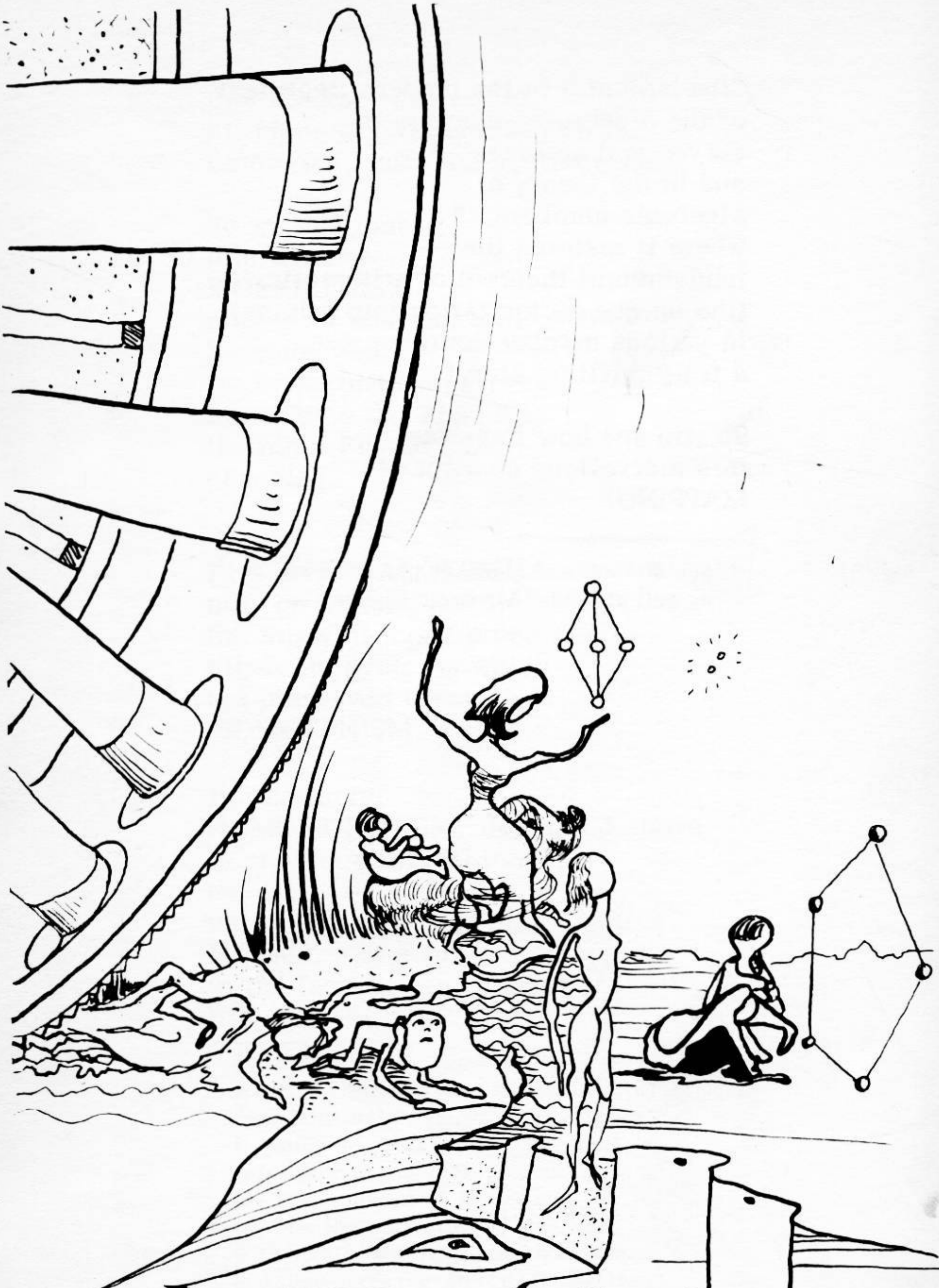
* For a very simple illustration
see E. V. Huntington:
"Fundamental Propositions of Algebra"
in which he gives (on pp. 156-165)
two concrete algebras,
(1) The Addition of Angles and
(2) The Multiplication of Distances,
with appropriate postulates for both ,
and then gives the ABSTRACT algebra
which includes BOTH of them
as well as ALL OTHERS which are
isomorphic with them!

"fundamental in the modern treatment
of the algebraic geometry of
curves and surfaces" *
and in the theory of
algebraic numbers, *
where it restores the
fundamental theorem of arithmetic
(the unique factorization into primes)
in various number domains —
a truly thrilling story!

So you see how important is
this marvellous concept of
MAPPING!

---

* See Birkhoff and MacLane pp. 360ff and 404ff as well as Reid: "Algebraic Numbers".



XVII. From POSET to BOOLEAN ALGEBRA.

We have seen that
if you start with a
POSET,
characterized by
the THREE properties (p. 76):

P1: $x \leqq x$ (reflexive)

P2: if $x \leqq y$ and $y \leqq x$, then $x = y$
(anti-symmetric)

P3: if $x \leqq y$ and $y \leqq z$, then $x \leqq z$
(transitive),

and then introduce
the operations $\frown$ and $\smile$,
with "closure" under
these two operations,
so that there is a
g.l.b. and l.u.b. for
EVERY pair of elements,
as discussed in Chapter XII.,
we thus get a
LATTICE.

Now there are
many DIFFERENT KINDS of LATTICES.
You remember that
if, to the BASIC postulates for a
GROUP,
you add another,
namely, the commutative law,
you get another kind of Group,
called a COMMUTATIVE Group
or an Abelian Group (p. 39).
Similarly
by adding various postulates to
those on p. 126,
we obtain

various kinds of Lattices, with important and useful properties.

$x \cap z = x$
$x \cup z = z$

For instance, if we add

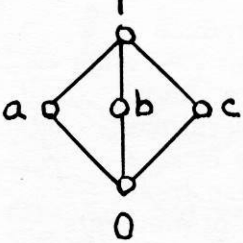
$$\text{L5:} \quad \left| \begin{array}{l} \text{If } x \leqq z, \text{ then} \\ x \smile (y \frown z) = (x \smile y) \frown z \ * \end{array} \right.$$

$(x \cup y) \cap (x \cup z) = (x \cap z) \cup (x \cap z)$

we get a MODULAR LATTICE, or a "Dedekind Lattice".

$(x \cup y) \cap z =$

Thus

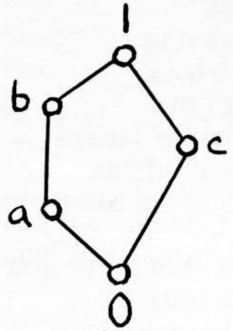


is MODULAR:
for example, take $a \leqq 1$,
then $a \smile (b \frown 1) = a \smile b = 1$ and
$(a \smile b) \frown 1 = 1 \frown 1 = 1$
so that
$a \smile (b \frown 1) = (a \smile b) \frown 1$,
and similarly for all other sets of three elements for which $x \leqq z$, as required by L5.

---

* Do not confuse this with L3, the associative laws, p. 126, which contain either all $\frown$ or all $\smile$, but NOT BOTH $\frown$ and $\smile$ as in L5.

Whereas



is NON-MODULAR:
thus if we choose $x = a$, $y = c$, $z = b$,
where $a \leqq b$ and yet
$a \smile (c \frown b) \neq (a \smile c) \frown b$,
for
$$c \frown b = 0,$$
and therefore
$a \smile (c \frown b) = a \smile 0 = a$;
whereas
$$(a \smile c) = 1$$
and therefore
$(a \smile c) \frown b = 1 \frown b = b$;
thus the results, a and b,
are NOT the same,
so that
$a \smile (c \frown b) \neq (a \smile c) \frown b$,
which proves that this lattice is
NON-MODULAR,
since it does not satisfy L5.
Indeed,
this is the SMALLEST non-modular lattice,
and is the ONLY
non-modular lattice having
FIVE elements.*

---

* SHOW that all the other lattices
having 5 elements (p. 116)
DO satisfy L5
for ALL x, y, z, for which $x \leqq z$,
and are therefore MODULAR.

And of course
any lattice having
a subset which is
ISOMORPHIC to
this non-modular lattice
is itself non-modular,
since not all of its elements
satisfy L5 —
which makes this particular
non-modular lattice
a useful instrument for
recognizing non-modularity in
a lattice having
a larger number of elements than 5,
as you will see subsequently.

By the way,
note that this lattice
has its
greatest and least elements
designated by 1 and 0, respectively.
This way of designating
the greatest (by 1) and
the least (by 0)
is quite customary;
and these symbols
do not of course
represent the NUMBERS 1 and 0,
but are merely shorthand for
"greatest" and "least".
Thus, in (18) on p. 93,
the "least" ("0") is the NUMBER 1,
and the "greatest" ("1") is
the NUMBER 36.
And,
in the infinite set on p. 104,
WITH THE NUMBER 0 INCLUDED,
we have
the "least" ("0") is again the NUMBER 1,
whereas the "greatest" ("1") is the
NUMBER 0!
And so on, in other lattices.

Further,
if we now add

L6': $x \frown (y \smile z) = (x \frown y) \smile (x \frown z)$

we get what is known as a
DISTRIBUTIVE LATTICE:
you doubtless recognize
this additional postulate
as being similar to
the well-known
Distributive Law of
ordinary algebra
discussed on p. 42.
You will remember that
the Distributive Law holds also in
Boolean Algebra (pp. 42ff),
and that there,
besides,
there is
ANOTHER Distributive Law.
Thus, in Boolean Algebra,
there are
TWO Distributive Laws,
namely:

$$\left.\begin{aligned} &x\,(y + z) = (xy) + (xz) \quad \text{and} \\ &x + (yz) = (x + y)\;(x + z) \end{aligned}\right\} \quad *$$

It is interesting to note that
from L6' one can PROVE
L6": $x \smile (y \frown z) = (x \smile y) \frown (x \smile z)$; **

---

* See pp. 43 and 44.

** $(x \smile y) \frown (x \smile z)$

$= [(x \smile y) \frown x] \smile [(x \smile y) \frown z]$ by L6'

$= x \smile [(x \smile y) \frown z]$ by L4 (p. 126)

$= x \smile [(x \frown z) \smile (y \frown z)]$ by L2 and L6'

$= [x \smile (x \frown z)] \smile (y \frown z)$ by L3

$= x \smile (y \frown z)$ by L4. Q.E.D.

in other words,
a Distributive Lattice has
TWO Distributive Laws.

Furthermore
L6' also implies

$$\text{L6: } (x \frown y) \smile (y \frown z) \smile (z \frown x) =$$

$$(x \smile y) \frown (y \smile z) \frown (z \smile x)$$

as well as L5 !

An interesting THEOREM here is:
EACH of the identities
L6, L6', L6"
implies the other two as well as L5! *

Thus, if a lattice is
Distributive,
it is automatically modular —
but NOT conversely,
since L5 does NOT imply
any of the three:
L6, L6', or L6".
Part of this theorem was proved
as late as 1936 by
K. Menger:
"New foundations of
projective and affine geometry",
Annals of Math. 37 (1936), pp. 456 - 82;
which shows
how new is this subject and
its new approach to
geometry!
But more of this later.

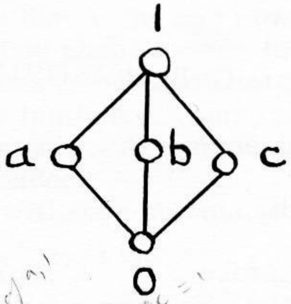
---

* G. Birkhoff: "Lattice Theory" (1948), p. 133.

Note that the two lattices on pp. 148, 149
(the first of which is modular and
the second non-modular)
are BOTH NON-DISTRIBUTIVE, *

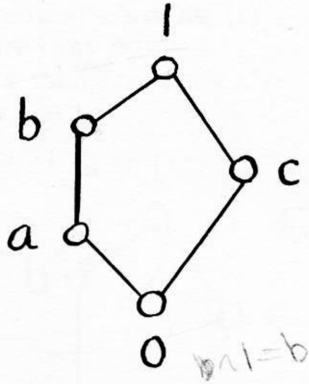
---

* Thus in



GLB of a, 1
lub of b & c = 1

$a \frown (b \smile c) = a \frown 1 = a$
whereas $(a \frown b) \smile (a \frown c) = o \smile o = o$
so that
$a \frown (b \smile c) \neq (a \frown b) \smile (a \frown c)$,
and this lattice is therefore
NOT distributive,
since it does not satisfy L6' on p. 151.
And similarly in



a ∪ 0 = a
b∩1=b

1 a 0

show that $b \frown (a \smile c) \neq (b \frown a) \smile (b \frown c)$
which makes this lattice also
NOT distributive.

and indeed,
any lattice which is
NOT distributive
always contains
one of these two
as a sublattice! —
a theorem due to G. Birkhoff (1934).

And now let us see
how to get a
Boolean Algebra
from a
Distributive Lattice,
as promised.

XVIII. More than One Boolean Algebra!

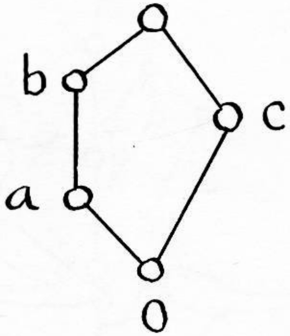
Note that at the end of
the last chapter,
I spoke of "a" Boolean Algebra.
This implies of course that
there are more than one,
and indeed
you will soon see many!

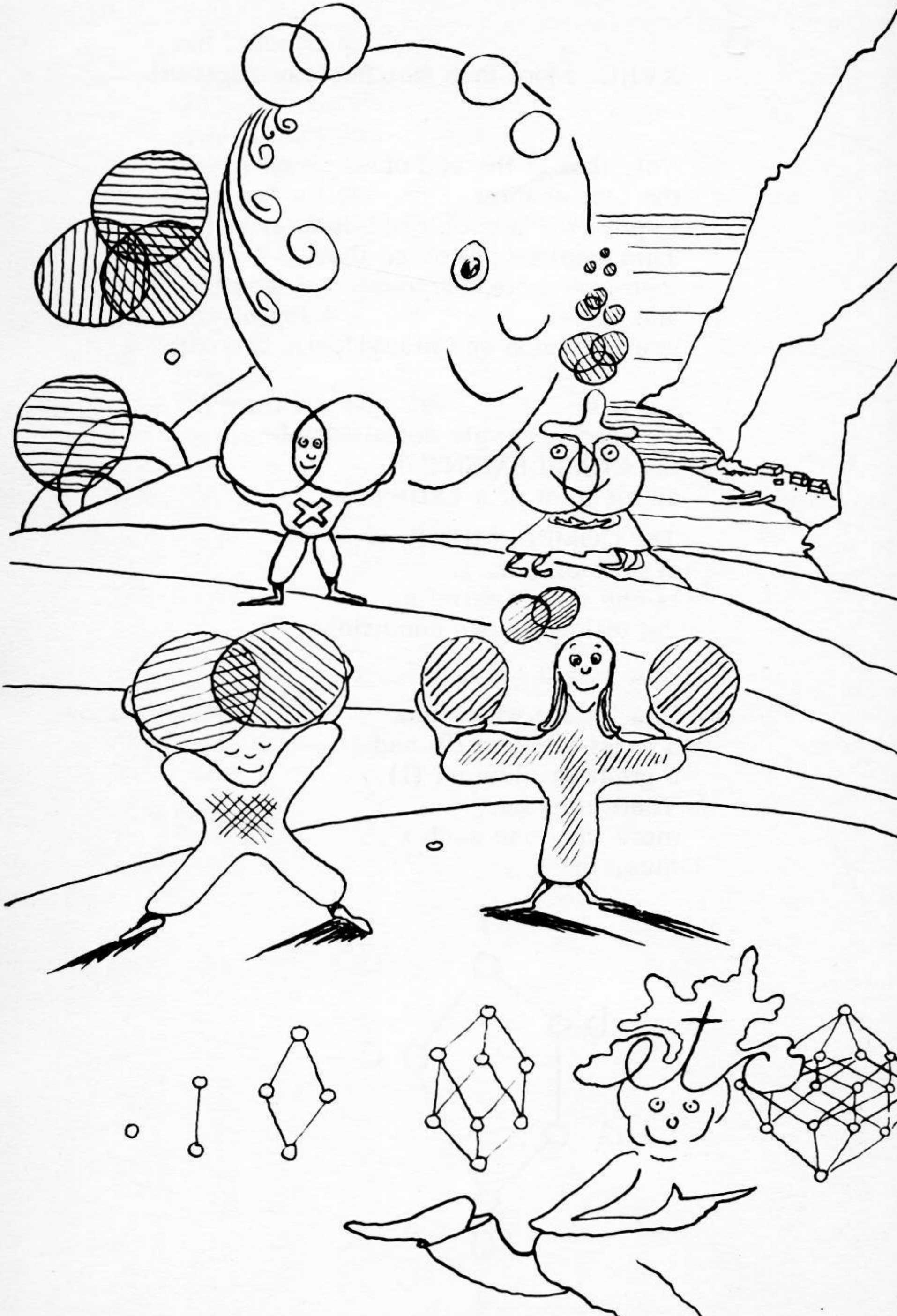
But first
we need a simple definition of
the COMPLEMENT of
an element of a Lattice:

The COMPLEMENT, x',
of an element, x,
is one which satisfies
the following two conditions:

$x \frown x' = 0$ and $x \smile x' = 1$

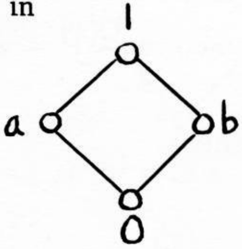
in a lattice which has
a least element (0) and
a greatest element (1).
There may be
more than one such x',
thus, in




etc

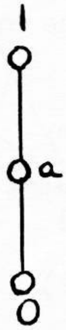
the element, a,
is a complement of c since
$a \frown c = 0$ and $a \smile c = 1$;
but so is b also
a complement of c since
$b \frown c = 0$ and $b \smile c = 1$.

Whereas in



each element has
one and only one
complement —
for example,
b is the complement of a because
$a \frown b = 0$ and $a \smile b = 1$,
and it is the ONLY complement of a.
(Test the others and see.)
Similarly a is the ONLY complement of b,
and 0 and 1 are
each other's
one and only one complement.

On the other hand,
in the lattice



the element a has
NO COMPLEMENT,
and similarly for
other CHAINS (see p. 82) * —
thus,
although $a \frown 0 = 0$
but $a \smile 0 \neq 1$ (since $a \smile 0 = a$);
and
though $a \smile 1 = 1$
but $a \frown 1 \neq 0$ (since $a \frown 1 = a$);

furthermore
a is not its own complement
since $a \frown a \neq 0$ and $a \smile a \neq 1$.
And so
the element a here has
NO COMPLEMENT.

Now,
a Lattice in which
EVERY element has
ONE OR MORE complements
is called
a COMPLEMENTED Lattice.
Note also that
in a Distributive Lattice,
each element has
AT MOST ONE complement. **
Consequently,
if a Lattice is
BOTH COMPLEMENTED AND DISTRIBUTIVE,
each element then has
a UNIQUE complement
(i.e. has ONE and ONLY one complement).
Such a lattice is called
a BOOLEAN ALGEBRA.

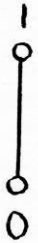
---

* Except of course a chain which has
exactly two elements, 0 and 1
(which are each other's complements).

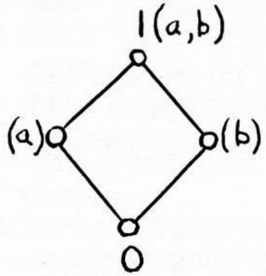
** For the PROOF of this,
see p. 269.

⌐ Following are
the simplest Boolean Algebras:

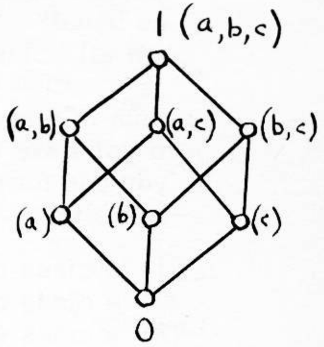
(1)



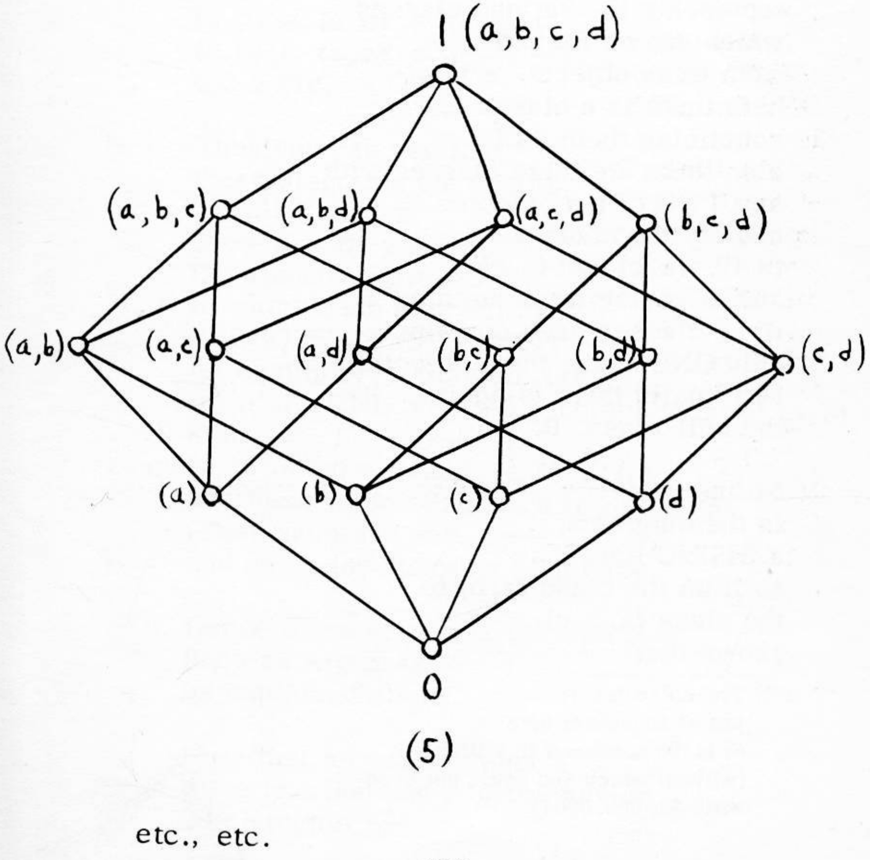
(2)



(3)


1 (a,b,c)
(a,b)
(a,c)
(b,c)
(a)
(b)
(c)
0

(4)


1 (a,b,c,d)
(a,b,c)
(a,b,d)
(a,c,d)
(b,c,d)
(a,b)
(a,c)
(a,d)
(b,c)
(b,d)
(c,d)
(a)
(b)
(c)
(d)
0

(5)

etc., etc.

Note that in all of these,
the elements
(represented by the little circles,
as usual)
are all "classes":

thus in diagram (3) on page 158,
if you have two objects, a and b,
you can form FOUR classes,
namely,

1) a class containing both a and b,
2) a class containing a only,
3) a class containing b only,
4) a class containing NEITHER! and called
   the NULL-CLASS. *

Similarly diagram (4)
represents the various classes
which can be formed
from three objects, a, b, c;
here there is a class, 1,
containing them ALL,
also there are three classes with
any TWO of the objects,
namely the classes
(a, b), (a, c) and (b, c);
and of course there are also
three classes each containing
only ONE of the three objects;
and finally there is again
the null-class, 0.

Notice that
in the diagrams,
a RISING line,
as from the class (a, b) to
the class (a, b, c)
shows that

---

* The null-class is
just as important here
as is the number 0 in arithmetic
(without which you could not even
write $1, 000, 000!)

the lower one (a, b) is
contained in the upper (a, b, c),
etc.
And the LEAST class that
contains BOTH a and b
is (a, b) which "covers" them both.

Thus $a \frown b = 0$ and $a \smile b = (a, b)$,
and similarly for
any other pair of elements ("classes") —
as, for example:
the LEAST class containing
the classes (a, b) and (a, c)
is of course the class (a, b, c),
and the GREATEST class which is
COMMON to both (a, b) and (a, c)
is of course the class (a),
that is:
$(a, b) \frown (a, c) = (a)$ and
$(a, b) \smile (a, c) = 1$,
etc., etc.

Thus you can see that
these diagrams on p. 158
are all Lattices.
Show that they are all
Distributive Lattices
in which each element has
a UNIQUE COMPLEMENT —
for example, in diagram (4),
(c) is the ONE and ONLY complement of (a, b)
since
$(a, b) \frown (c) = 0$ and $(a, b) \smile (c) = 1$ (a, b, c),
and there is no other such complement
(Test them and see.) —
and so on for the other elements.

Hence these diagrams all represent
Boolean Algebras,
as explained above.

Note that for each integer n
there is a Boolean Algebra having
2^n elements —

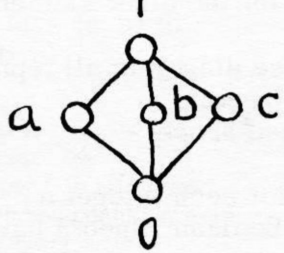
thus for $n = 1$, $2^n = 2^1 = 2$ which
is the number of elements in
diagram (2) on page 158;
for $n = 2$, we have $2^n = 2^2 = 4$,
represented by diagram (3) with its
FOUR elements;
similarly diagram (4) shows
a Boolean Algebra having
$n = 3$ and $2^n = 2^3 = 8$ elements;
and in diagram (5)
we have
$n = 4$ and $2^n = 2^4 = 16$ elements,
etc., etc.

And so you can see that
there is an infinite number of
Boolean Algebras,
satisfying the condition that
each is a
Distributive Lattice in which
each element has
a UNIQUE COMPLEMENT.

The VERY simple Boolean Algebra in
diagram (2) on p. 158
is of the utmost PRACTICAL importance,
as we shall soon see.

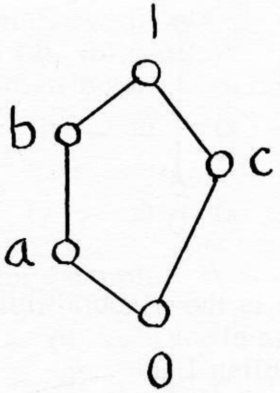
But first, for contrast,
let me recall to you
a few important lattices which
do NOT represent Boolean Algebras:

(1)



Although this lattice is MODULAR (p. 148)
and COMPLEMENTED
(every element has at least one
COMPLEMENT —
try it and see!),
still it is NOT a Distributive Lattice
as explained on p. 153,
and hence NOT a Boolean Algebra (p. 157),
although it looks so much like
diagram (3) on p. 158
which IS a Boolean Algebra,
as explained there.

(2)



This lattice is, as you know,
NON-MODULAR (p. 149) and
NON-DISTRIBUTIVE (p. 153)
and so
it, too,
is NOT a Boolean Algebra.

(3) A CHAIN, though it is
DISTRIBUTIVE,
is NOT COMPLEMENTED (p. 157)
and is therefore also
NOT a Boolean Algebra,
except only

the chain containing
exactly two elements, 0 and 1,
which are each other's complements —
this simplest chain,
represented by diagram **(2)** on p. 158
IS, as you know,
a Boolean Algebra.

And so on.

Let us now look
a little more carefully
at this VERY simple Boolean Algebra,
represented by:

for this is the Algebra which
gives us all of
Aristotelian Logic
in a nutshell! *

It is easy to see that

(1) there are only TWO elements here, namely, 0 and 1;

(2) the only possible combinations under $\frown$ and $\smile$ are:

(a) $0 \frown 0 = 0$
(b) $0 \frown 1 = 1 \frown 0 = 0$
(c) $1 \frown 1 = 1$

and

---

* See "Mits, Wits and Logic", in this series.

(d) $0 \smile 0 = 0$
(e) $0 \smile 1 = 1 \smile 0 = 1$
(f) $1 \smile 1 = 1$,

so this system has
"closure" for both $\frown$ and $\smile$ (p. 37);

(3) when 1 is operated on by $\frown$ with
any other element (here 0 or 1)
it leaves that element unchanged,
as in (2), (b) and (c) above;
(4) when 0 is operated on by $\smile$ with
any other element (0 or 1),
it leaves that element unchanged,
as in (2), (d) and (e) above;
(5) both $\frown$ and $\smile$ obey the
commutative laws,
as in (2), (b) and (e);
(6) both distributive laws hold:

$$x \frown (y \smile z) = (x \frown y) \smile (x \frown z)$$

and

$$x \smile (y \frown z) = (x \smile y) \frown (x \smile z),$$

as, for example,

$$\begin{cases} 1 \frown (0 \smile 1) = 1 \frown 1 = 1 \quad \text{and} \\ (1 \frown 0) \smile (1 \frown 1) = 0 \smile 1 = 1; \end{cases}$$

AND

$$\begin{cases} 1 \smile (0 \frown 1) = 1 \smile 0 = 1 \quad \text{and} \\ (1 \smile 0) \frown (1 \smile 1) = 1 \frown 1 = 1; \end{cases}$$

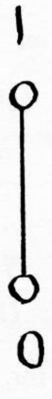
and similarly for
any other combinations of
x, y, and z;

(7) $x \smile x' = 1$
(where x' is the complement of x),

since 1 is the complement of 0,
and 0 is the complement of 1
(see (2), (e));

(8) $x \frown x' = 0$
(see (2), (b).

If you now compare
these simple statements with
the postulates for
the Boolean Algebra* which
applies to
Aristotelian Logic,
you will see that
they are IDENTICAL,
and you will surely admit that
this Lattice approach is
even simpler and more vivid,
all the above-mentioned postulates
being embodied in
the amazing little diagram:



And of course
all the applications to
electric circuit theory, **
etc., etc.,

---

* See "Mits, Wits and Logic", pp. 187ff.

** See F.E. Hohn: "Some mathematical Aspects of Switching." Am. Math. Monthly: 62(2), pp. 79 - 90.

are in no way disturbed,
since they are
ISOMORPHIC with
this little lattice!

XIX. Applications to Group Theory.

To fully appreciate this section,
one must read it with
paper and pencil in hand,
for,
only by actually
"playing the game"
can it be properly ENJOYED!
If the reader does not wish to do this,
he can of course
turn to pp. 174ff
to see the final result,
BUT
it would be like
turning to the last pages of a
"Whodunnit",
and really miss the fun!

As was mentioned on p. 15,
the elements of a group
may be
SUBSTITUTIONS —
for example
(123) means that
1 is to be replaced by 2
2 by 3 and
3 by 1,
as in the expression

$$x_1x_2 + x_3$$

which then becomes

$$x_2x_3 + x_1 ,$$

and so on.

Now, as on p. 27,
when one such substitution is

FOLLOWED by another,
say, (13)
(which changes 1 to 3 and 3 to 1),
we say that
the first SUBSTITUTION has been
MULTIPLIED by the second one,
and this is written

(123) (13).

The given expression then becomes

$$x_2x_1 + x_3.$$

Note that this same final result
would have been obtained by
applying the
SINGLE SUBSTITUTION (12)
to the original expression

$$x_1x_2 + x_3.$$

Hence we say that
the PRODUCT of (123) and (13) is (12),
thus

$$(123)\,(13) = (12).$$

Incidentally note that
"multiplication" here is
NOT commutative, since

$$(13)\,(123) = (23) \neq (12) \text{ !! *}$$

And so it is important to say here
whether you are
multiplying an element, a, by
an element, b,
ON the RIGHT or
ON the LEFT,
since ab is not necessarily equal to ba,
as mentioned above.

---

* See "Galois and the Theory of Groups",
in this series,
for a more detailed discussion
of this and other details
referred to on the following pages.

Now it is obvious that
for three numbers, 1, 2, 3,
there are SIX substitutions possible,
namely,
the SIX permutations of
these three numbers,

(a) two at a time:
    (12), (13), (23) or

(b) all three at a time:
    (123), (132),
    and of course

(c) the substitution
    (1) (2) (3) or I
    which leaves every expression
    unchanged,
    and is called the
    IDENTITY substitution. (see p. 49).

These six substitutions form
a GROUP,
since they satisfy
the four postulates for a group on p. 48.
The reader will enjoy
verifying this statement —
it is much more fun than
a cross-word puzzle!

Furthermore,
SOME of the six substitutions form
a Group among themselves,
for instance:

I, (12);

also I, (13) and I, (23);

and I, (123), (132).

This too should be checked by
the reader,
with the aid of p. 48,
if he really wishes to
have some fun with all this!

These smaller groups within
the original group
are quite naturally called
SUB-groups of the given group.
Now, among these subgroups,
one, the last one mentioned above,
namely the one consisting of
the elements

$$I, (123), (132)$$

is a very special kind of
Subgroup,
known as an
INVARIANT (or NORMAL) subgroup —
so named because
it remains unchanged when
each element in it is
"transformed" by all the elements of
the original group.

To make this clear
let us first define the meaning of
the term "transform",
as used in this connection:

To find the TRANSFORM of
an element, a, by an element, b,
one must
MULTIPLY a by b on the RIGHT
and by the inverse of b
(call it b')
on the LEFT,
so that

$$b'a\,b$$

is the TRANSFORM of a by b.

Now to show that
the subgroup I, (123), (132)
is INVARIANT,
we shall have to
find the TRANSFORMS of
all its elements by
all the elements of

the original group of
six elements (p. 169).
For instance,
to find the TRANSFORM of

(123) by (12),

proceed as follows:

(12) (123) (12) = (132) *

(Note that (12) is the inverse of (12)
since (12) (12) = I)!

And observe that the result,

(132)

IS IN OUR SUBGROUP.
Similarly,
find the remaining TRANSFORMS
and you will find that
you always get
either I or (123) or (132),
and consequently
our subgroup
IS an INVARIANT SUBGROUP,
since all our transforms are in it,
so that the subgroup has remained
UNCHANGED.
Whereas,
it is easy to show that
the subgroup consisting of
the elements I and (12)
is NOT INVARIANT —
thus
the TRANSFORM of
(12) by (123) is

---

* To get this result,
either perform the actual substitutions
in succession,
on some expression, as on pp. 167 and 168;
or look up the short-cut in
"Galois and the Theory of Groups",
previously referred to.

(132) (12) (123) *

or (23) which
is NOT in this subgroup,
and so
this particular subgroup
is NOT an INVARIANT one.

Now how does all this
connect up with
Lattice Theory?

Let us see!

Consider the group of the
4! (= 4x3x2x1 = 24)
permutations of the
four numbers, 1, 2, 3, 4,
like
(12), (234), (1234), (23), (12)(34), I, etc.
Of these you will find that
twelve of the twenty-four
leave the expression

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)^{**}$$

unchanged in value —

as, for example,
(123) changes the above expression to

$$\underline{(x_2 - x_3)}\ (x_2 - x_1)\ \underline{(x_2 - x_4)}\ (x_3 - x_1)\ \underline{(x_3 - x_4)}\ \underline{(x_1 - x_4)}$$

in which
the four factors which are
UNDERLINED
are the same as before except for
the order in which they occur,

---

* Since (132) is the inverse of (123).

** This is a very useful type of
expression, in this connection —
see "Galois and the Theory of Groups".

and the remaining two factors
are merely changed in sign
since $(x_1 - x_2) = -(x_2 - x_1)$
and $(x_1 - x_3) = -(x_3 - x_1)$,
and of course two such
changes in sign
bring the expression back to
its original value.

Similarly you will find that
there are twelve elements here
(of the original 24)
which have this property of
leaving the expression (on p. 172)
unchanged in value.
These 12 elements
constitute a group among themselves,
known as
the Alternating Group of
the permutations on
four numbers,
and is designated by

A_4.

The above-mentioned 12 elements in A_4 are:
(123), (132), (124), (142), (134), (143), (234),
(243), (12) (34), (13) (24), (14) (23), and I.

The reader may wish to show that
this set of elements
actually is a Group,
satisfying the requirements on p. 48.
Now this subgroup
itself has other subgroups,
as, for example,

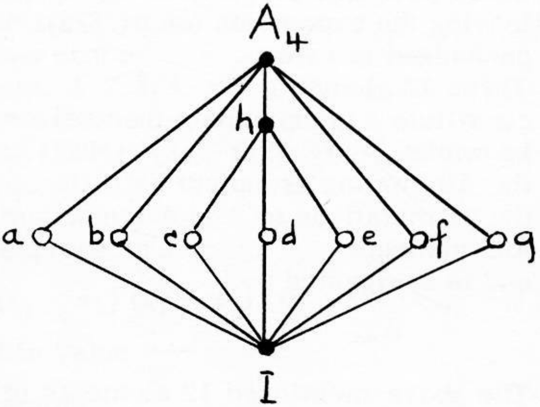
I, (123), (132),

and many others,
but not "ad infinitum",

since it is a finite group and
the smallest subgroup is of course
I itself.

Now some of these subgroups are
INVARIANT (NORMAL) and
others are not.
And it is interesting to see
the following
TWO LATTICES which
belong to A_4 :

(1)



representing
ALL the subgroups of A_4, namely:

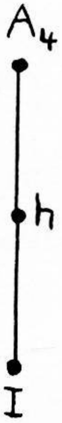
a = I, (123), (132)
b = I, (124), (142)
f = I, (134), (143)
g = I, (234), (243)
c = I, (12) (34)
d = I, (13) (24)
e = I, (14) (23)
h = I, (12) (34), (13) (24), (14) (23)

which,
together with
I, the smallest subgroup, and
A_4, the all-inclusive one,
containing all 12 elements,
give a picture of
the LATTICE of
ALL the subgroups of A_4 .

In general,
such a lattice of any group, G,
is designated by L(G).

AND

(2)



shows ONLY * the
INVARIANT (NORMAL) subgroups of A_4 .

---

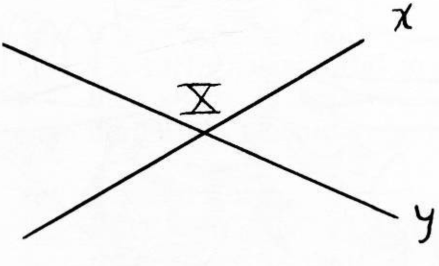
[*] Whereas the subgroup a (see p. 174)
was shown to be INVARIANT in
the group of SIX elements on pp. 170, 171,
it is NOT INVARIANT in A_4:
thus show that the transform of (123)
by (12) (34) is (142),
and is therefore NOT in subgroup a.

These lattices of a group
characterize the structure of
the group,
and thus facilitate the
study of groups,
just as
Lattice Theory has been shown (pp. 155ff)
to throw light on the study of
Boolean Algebras,
and,
in subsequent chapters
will be shown also
to be a valuable tool in
the modern study of
Geometries.

XX. Applications to Geometry.

Let us consider next
another application of
Lattice Theory:

Take as elements
the points, lines, and planes of
ordinary space,
with the relationship of
INCLUSION (p. 82),
in the sense that
points are included in lines,
lines in planes,
and planes in space.
Here $x \frown y$ is
the part common to x and y
(as with classes),
thus:
if x and y are two intersecting lines,
then $x \frown y$ is
their POINT of intersection, X:



and $x \smile y$ is
the LEAST element which
contains both x and y,
namely, in this case,
the PLANE determined by them;
and if the lines are parallel,

then $x \frown y = 0$ and $x \smile y$ is again
the plane determined by them.

If we now designate
the null-class by 0,
and three-dimensional space by 1,
we have a lattice,
for:

(1) the part COMMON to
ANY two elements,
their $\frown$,
is itself an element of the system:
thus
    (a) if x and y are POINTS,
    then $x \frown y = 0$;

    (b) if they are PARALLEL LINES,
    again $x \frown y = 0$;

    (c) if INTERSECTING LINES,
    $x \frown y = X$ which is
    their POINT of intersection,
    as mentioned above;

    (d) if x is a PLANE and
    y a LINE not parallel to x,
    then $x \frown y$ is again
    their POINT of intersection;

    (e) if x and y are
    INTERSECTING PLANES,
    then $x \frown y$ is their
    LINE of intersection;
and, in general,
the $\frown$ of ANY two elements
is itself an element of the system.

Furthermore
(2) there is an element of the system
which is

the LEAST one CONTAINING
ANY TWO given elements,
namely, their $\smile$;

for instance:

(a) if x and y are POINTS
then $x \smile y$ is
the straight LINE determined by them;

(b) if they are INTERSECTING LINES,
or PARALLEL LINES,
then $x \smile y$ is
the PLANE determined by them;

(c) if they are SKEW LINES,
then $x \smile y = 1$ (space itself);

(d) if x is a LINE and
y a POINT NOT ON the line,
then $x \smile y$ is again
the PLANE determined by them;

and so on, in general,
for ANY x and y,
$x \smile y$ is an element of the system.

And so,
this is a Lattice (p. 112).

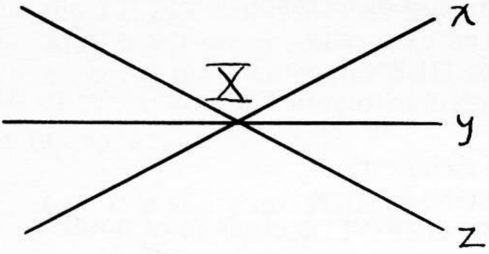
Let us now see
what kind of lattice it is.

At first glance
it might seem that it is
the same as
the Algebra of Classes (see p. 83),
since we have here really
"classes" of points
(namely,
straight lines, planes, space)
related by INCLUSION.
BUT
there is this DIFFERENCE:

in the Algebra of Classes
we have as elements
ALL POSSIBLE classes of
the basic constituents ("points") — p. 158,
whereas
here, in Geometry,
we have ONLY
CERTAIN DEFINITE classes of points,
namely,
straight lines, planes, and space:
thus we do NOT have
sets of THREE points like

among the elements here,
since such a set is
NEITHER a POINT,
NOR a STRAIGHT LINE,
NOR a PLANE,
NOR SPACE itself —
nor of course the null-class.

Now, it turns out that
the Lattice here,
UNLIKE the Algebra of Classes,
is NOT a Boolean Algebra,
for it does NOT satisfy
the two distributive laws (p. 151):
thus, consider
three straight lines x, y, z,
all meeting in the same point, X:



and let us represent
the plane of this page by
the letter p.
We shall now show that

$$x \frown (y \smile z) \neq (x \frown y) \smile (x \frown z),$$

that is,
the Distributive Law (p. 151)
does NOT hold here:
for

$$x \frown (y \smile z) = x \frown p = x$$

whereas

$$(x \frown y) \smile (x \frown z) = X \smile X = X$$

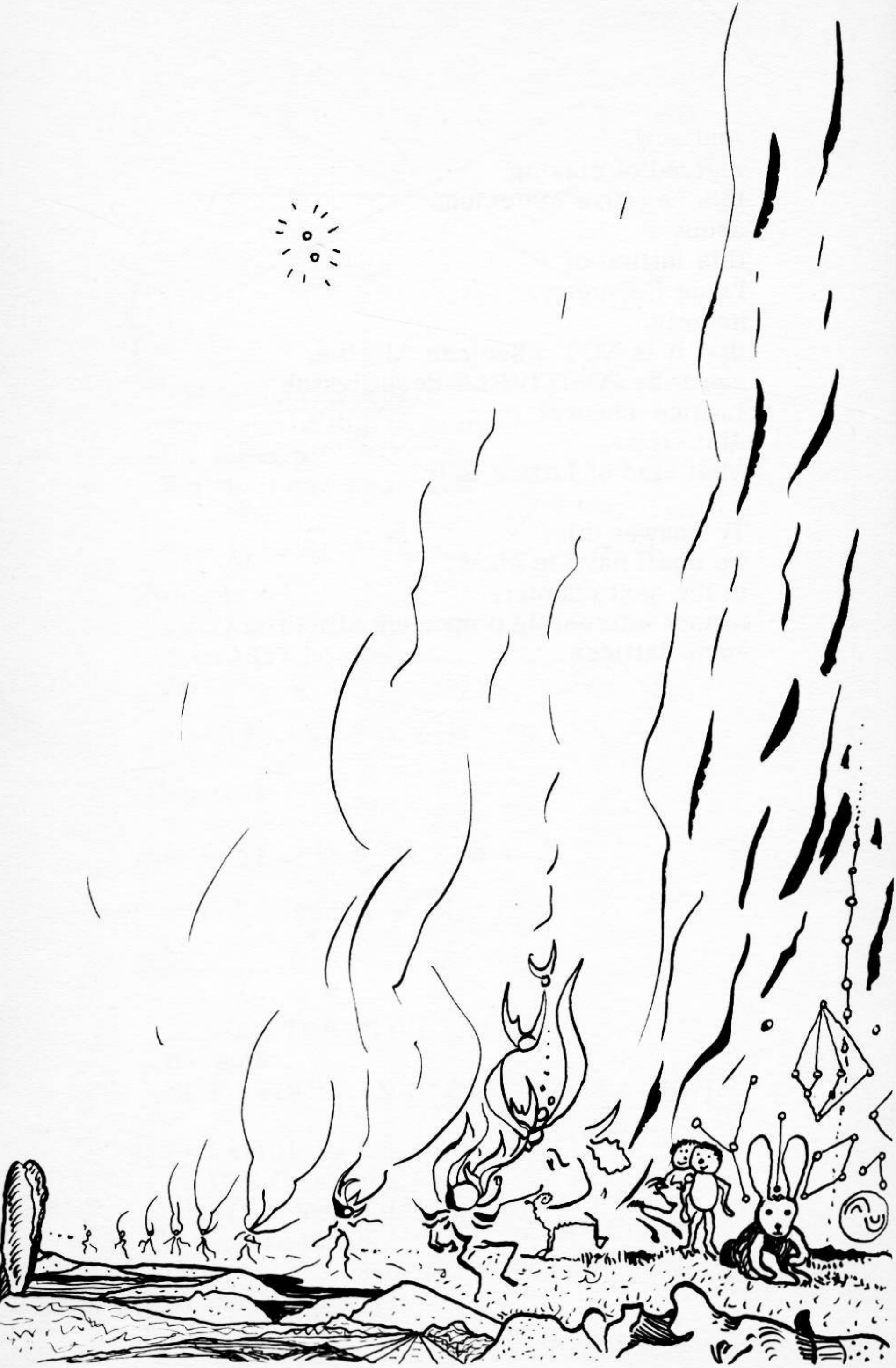
and of course $x \neq X$.

And so
this lattice is
NOT DISTRIBUTIVE and
therefore
NOT a BOOLEAN ALGEBRA (see p. 157).

And similarly for
PLANE Geometry,
where we would designate
the entire plane
(rather than space)
by the element 1.

And now,
instead of making
this negative statement
about
this lattice of
Plane Geometry,
namely,
that it is NOT a Boolean Algebra,
can it be POSITIVELY described in
Lattice Theory?
And, if so,
what kind of lattice is it?

To answer this,
we shall have to show,
in the next chapter,
further interesting properties of
some lattices.

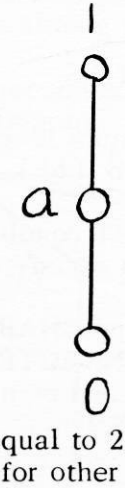


XXI. A Few Useful Lattices.

We have already seen
what is meant by
a MODULAR LATTICE (p. 148).
We shall now need to know
what is a
SEMI-MODULAR LATTICE —
one of the important concepts
introduced by
Garrett Birkhoff
and hence also known as
a "Birkhoff Lattice".

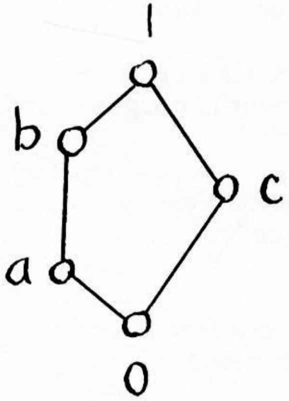
To do this
we must first give a few
very simple definitions:

(1) the LENGTH of a CHAIN (see P4 on p. 82)
of any finite number, n, of elements
is n − 1 :
thus, the chain having 3 elements



has a length equal to 2,
and similarly for other
finite values of n.

(2) the DIMENSION (or HEIGHT), $d\left[x\right]$,
of an element x of a poset, P,
is the maximum length of
all the chains having x as
greatest element:
thus, in the following poset



1 is connected to 0 by
two different chains —
0ab1 and 0c1,
having the "lengths" 3 and 2,
respectively,
and therefore
the "height" of "1"
is 3, since
3 is the maximum length of
the chains from 0 to 1.

An IMPORTANT condition that
SOME lattices satisfy is
the so-called
Jordan-Dedekind CHAIN CONDITION:
"All finite CONNECTED chains
between fixed end points
have the SAME length." *

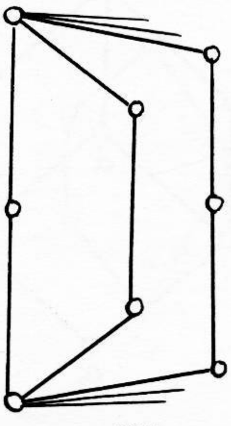
---

* See G. Birkhoff: "Lattice Theory", p. 11.

And you should know that
a CONNECTED chain is
one in which
each element "covers" the next (p. 99):
thus:
the lattices (5), (8), (9), etc. on p. 116
satisfy the J-D condition
whereas
(7) and (12) do NOT.

(3) similarly,
the dimension or length, $d\left[P\right]$,
of the poset, P, itself,
is the maximum length of
a chain in P:
thus $d\left[P\right]$ for the poset
illustrated on p. 184
is also 3.

It is interesting to note here
that
even though ALL the chains
of a poset
may be finite,
and have different lengths,
still
there MAY NOT be a
MAXIMUM one among them!
Thus,
in the poset represented by
the following diagram

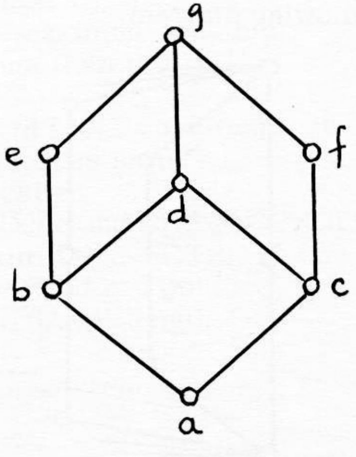


the chains have, respectively,
lengths 2, 3, 4, ... n − 1,...,
EACH of which is finite,
and yet there is
NO MAXIMUM one among them
(since there is no "largest" integer) —
and so,
this poset is NOT of FINITE length
(see (3) above).

(Incidentally,
it is obvious that
it does NOT satisfy the
J-D chain condition.)

Now,
a lattice of finite length is called
SEMI-MODULAR
IF and ONLY IF
its elements satisfy the
following condition:

If x and y cover a, and $x \neq y$,
then $x \smile y$ covers x and y.
For example, in

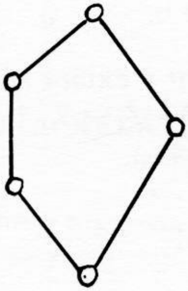


b and c both cover a,
and $b \smile c = d$, which
covers both b and c;
similarly,
for ANY two elements that cover
a third one here,
(e and d cover b,
d and f cover c)
it is true that
their $\smile$
covers the two elements
($e \smile d = g$ which covers e and d,
$d \smile f = g$ which covers d and f) —
thus the condition
mentioned above
IS satisfied here,
and this lattice therefore IS
a SEMI-MODULAR LATTICE.

Note, however, that
it is NOT a MODULAR lattice
since
if you consider the SUB-SET
consisting of the elements

a, b, e, g, f,

it can be easily shown that
this SUBSET is
ISOMORPHIC with the familiar
NON-MODULAR lattice



and therefore
the lattice on p. 186 is
itself NON-MODULAR
(see p. 150).

So you see that
a lattice may be
SEMI-MODULAR without being
MODULAR.*

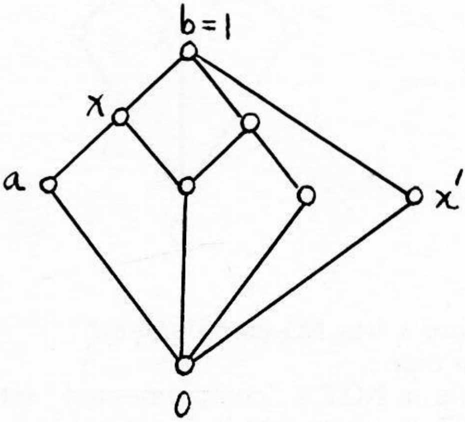
Now what has all this
to do with
GEOMETRY?

Well,
to show the connection,
we need
one more definition —
namely,
the meaning of
a "relatively complemented" lattice.
You already know that
a "complemented" lattice is
one in which
every element x
has at least one "complement", x' (pp: 157ff),
such that
$x \frown x' = 0$ and $x \smile x' = 1$.
And now
a "relatively complemented" lattice
is one in which:
for ALL a and b,
if $a \leqq x \leqq b$,
then an element y exists such that
$x \frown y = a$ and $x \smile y = b$.

---

* The converse, however,
is NOT true —
that is,
a MODULAR lattice
is ALWAYS
SEMI-MODULAR
(see p. 199).

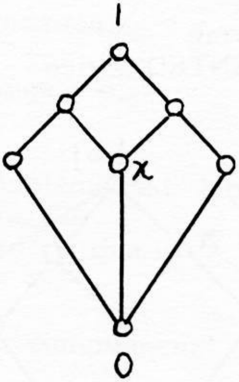
Now, the following
THEOREM is obvious:
if a lattice is
"relatively complemented"
it is automatically "complemented",
since for $a = 0$ and $b = 1$
we have
$x \frown y = 0$ and $x \smile y = 1$,
which makes y the complement of x.
BUT, the CONVERSE is
NOT necessarily true,
that is,
a lattice may be "complemented"
WITHOUT being "relatively complemented":
for instance,
in the following
COMPLEMENTED lattice



x has NO complement RELATIVELY to
a and b;
that is,
there is NO element y such that
$x \frown y = a$ and $x \smile y = b$ —
try it and see!
However,

x does have a COMPLEMENT x'
since $x \frown x' = 0$ and $x \smile x' = 1$,
and this is true also for
ALL the elements here.
Thus we have here
a lattice which IS "complemented",
but is NOT "relatively complemented".

Also, it is possible to have
a lattice which is
NEITHER "complemented"
NOR "relatively complemented",
thus:



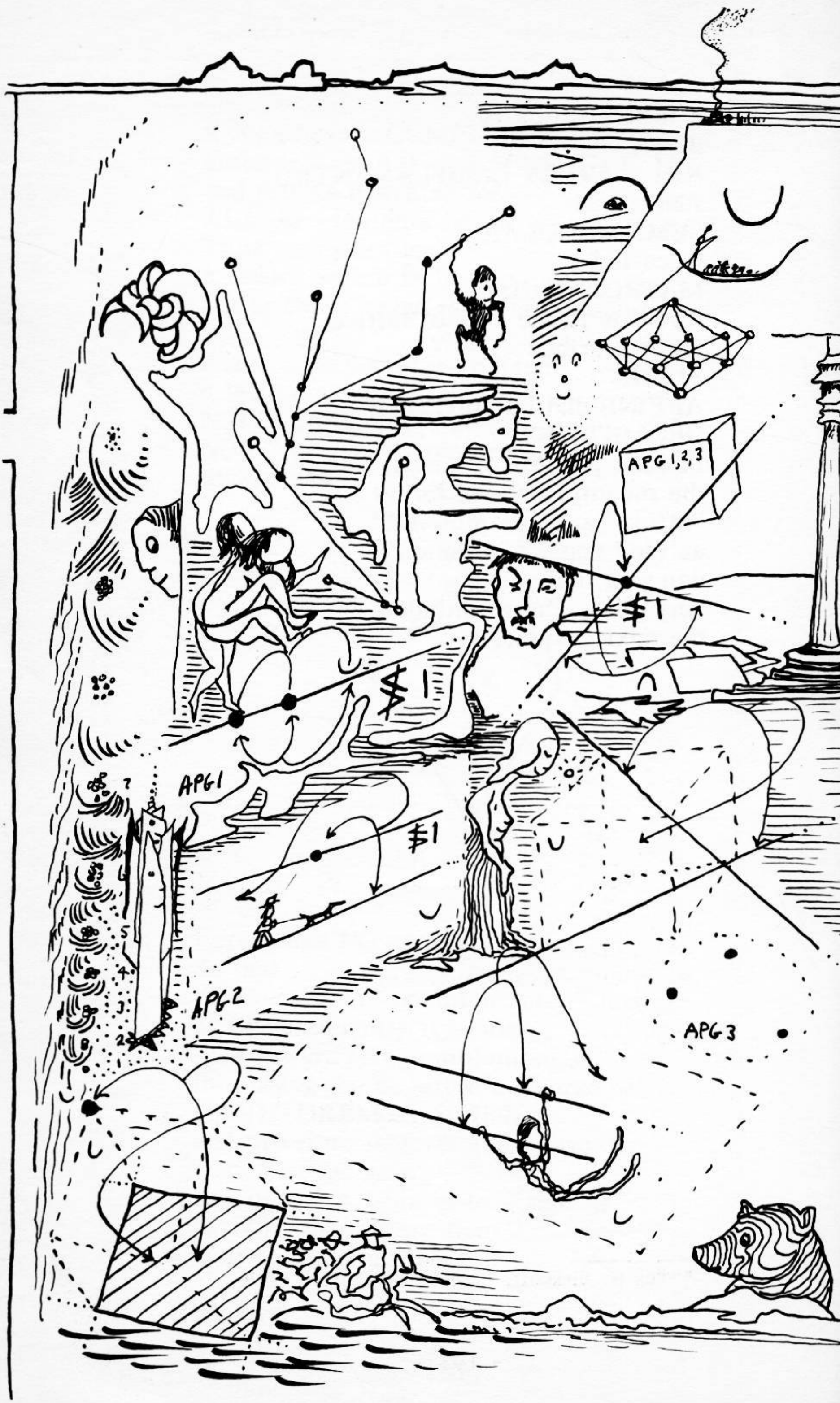
Here x has NO complement!
So that
this is NOT a "complemented" lattice
and consequently it is also
NOT "relatively complemented" —
this being the negative converse of
the THEOREM on p. 189,
and therefore follows automatically
from that theorem.
And of course this is true also of
any CHAIN of more than 2 elements
(see pp. 156 and 157).

Finally,

a lattice which is
BOTH
RELATIVELY COMPLEMENTED
AND
SEMI-MODULAR
is called a
MATROID lattice.
And this is the kind of lattice
that describes
PLANE
AFFINE and PROJECTIVE
GEOMETRIES.
In case you do not recall
the meanings of the terms
"affine" and "projective"
as they apply to geometry,
you will find
very neat, short reminders in
the next chapter. *

---

* See G. Birkhoff: "Lattice Theory", p. 110.


APG 1,2,3
$1
APG1
$1
$1
7
5
4
3
2
APG2
APG3

XXII. Affine and Projective Geometries.

Remembering the interpretation of
$\frown$ and $\smile$ in Geometry (p. 177)
you will appreciate
the following THREE postulates for
Affine Plane Geometry:

APG1: Any two distinct points p, q
are on one and only one line $p \smile q$ (p. 179).

APG2: Given line $p \smile q$ and
r, a point NOT on $p \smile q$,
there exists just one line through r,
namely, $r \smile s$, such that
$p \smile q$ and $r \smile s$, have NO common points
(which you will recognize as
the Euclidean parallel postulate).

APG3: There are three points not on a line.

And, further,
every plane AFFINE geometry
can be extended to
a plane PROJECTIVE geometry
by adding a "line at infinity", λ_∞ .
And so
for a PROJECTIVE geometry
we must replace APG2 above by

PG2': EVERY pair of distinct lines
has just one point in common;
thus
two finite lines ALWAYS intersect,
either on the finite plane
or (if "parallel") on λ_∞ .

AND

PG2": Every line contains
THREE or more points.

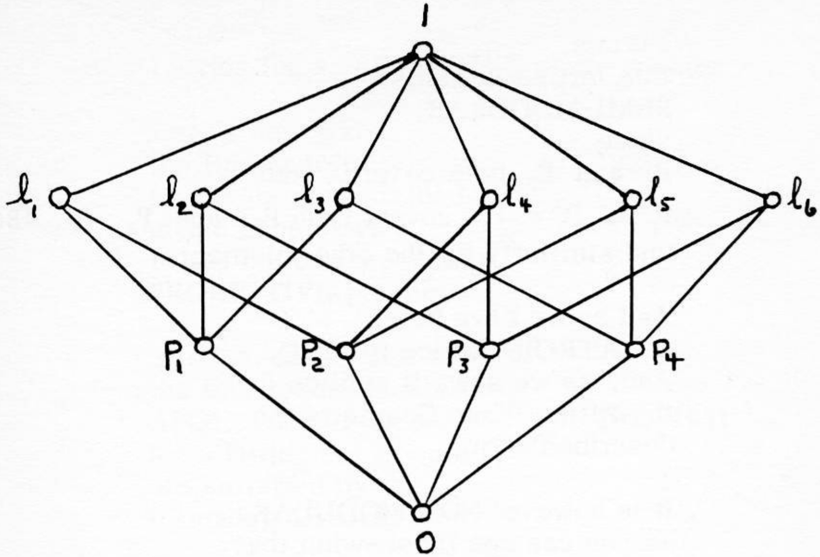
Thus
a set of points and lines
satisfying
APG1, PG2', PG2", APG3
is called
a plane Projective Geometry.

You may wonder why the expression
"every" plane affine geometry,
implying that there is
more than one such,
instead of just
the old familiar
Euclidean Geometry
which of course satisfies
APG1, 2, 3 on p. 192.
But you can easily see that
whereas Euclidean Geometry has
an infinite number of points,
the three postulates APG1, 2, 3
say nothing about
the total number of points
and therefore do not demand
that this be infinite.
In other words,
APG1, 2, 3 may describe
a plane affine geometry which
has only a finite number of points,
for example, four:

A. Lattice for an AFFINE plane geometry
having 4 points:
For this finite geometry
we can easily draw a diagram
representing its lattice
and satisfying APG1, 2, 3,
as follows: *

---

* Contrast this with diagram (5) on p. 158
which is the Boolean Algebra having
4 basic "points", a, b, c, d
(see p. 103 for this use of the word "points").



where P_i ($i = 1, 2, 3, 4$) are
the four points given,
l_j ($j = 1$ to 6) are

the six lines
each of which is determined
by two of the points,
thus $P_1 \smile P_2 = l_1$,

$P_1 \smile P_3 = l_2$, etc.;

0 is the null class,
and 1 is the plane containing
all these points and lines.

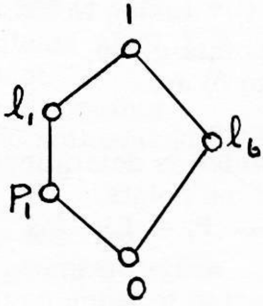
Now it is easy to see
that this is a
RELATIVELY COMPLEMENTED lattice:
Thus

$0 \leqq P_1 \leqq l_1$ and
there exists an element, P_2, such that
$P_1 \frown P_2 = 0$ and $P_1 \smile P_2 = l_1$ (p. 188),
and similarly for the other elements.

Further,
this lattice is also
SEMI-MODULAR,
since
P_1 and P_2 both cover 0, and
$P_1 \smile P_2 = l_1$ covers both P_1 and P_2 (p. 186),
and similarly for the other elements.

And so we have here
a MATROID lattice (p. 191).
And, as we saw, it applies to
the Affine Plane Geometry
described above.

It is however NOT MODULAR,
as you can see by showing that
the subset $0, P_1, l_1, 1, l_6$



is isomorphic with the
already familiar NON-MODULAR lattice
on p. 149.

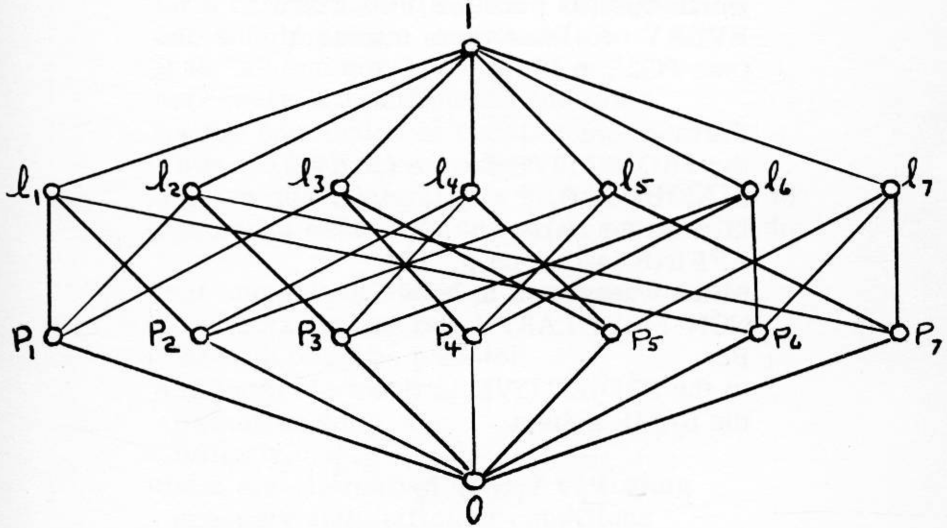
Indeed,
the lattice describing
ANY Affine Plane Geometry
may also be shown to be a
NON-MODULAR MATROID LATTICE.

Let us next examine a lattice for
a PROJECTIVE plane geometry:

B. Lattice for a PROJECTIVE plane geometry.

Let us now add
the line at infinity, λ_∞ (p. 192)
to the above-described
affine geometry,
to get the corresponding
PROJECTIVE geometry.

You can easily see that
the four requirements
APG1, PG2', PG2", APG3 (pp. 192, 193)
for a Projective Geometry
are satisfied by the
following Lattice:



where l_7 is the Line at Infinity, λ_∞ , and
P_7 is the point of intersection,
AT INFINITY,
of lines l_2 and l_5 .

Here the three PAIRS of lines,

l_1 and l_6
l_2 and l_5
l_3 and l_4

meet at the points P_6, P_7, P_5,
respectively,
which are all on the line $\lambda_{\infty}(l_7)$.

And this is in contrast to
the situation in the lattice on p. 194,
in which
these three pairs of lines
have NO point in common
and are therefore parallel there.
This of course emphasizes
the DIFFERENCE between
AFFINE Geometry,
which HAS parallel lines
(as demanded by APG2, p. 192)
and
PROJECTIVE Geometry,
which has NO parallel lines since
EVERY two lines there intersect
(see PG2', p. 192).

Furthermore,
the PROJECTIVE lattice (p. 196),
IS MODULAR, *
in contrast to the
AFFINE lattice on p. 194
which was shown to be
NON-MODULAR.
For
in the PROJECTIVE lattice
the SAME subset

$$0, P_1, l_1, 1, l_6$$

is NOT isomorphic with
the non-modular lattice on p. 195
since now $l_1 \frown l_6 \neq 0$, but
is equal to P_6 instead.

And so, whereas
the Projective Geometry Lattice

---

* And is therefore automatically
SEMI-MODULAR —
see footnote on p. 188.

IS modular, and
the Affine Geometry Lattice
is NOT modular —
which emphasizes
the DIFFERENCE between them —
still
since BOTH of them are
SEMI-MODULAR (see pp. 195 and 197),
as well as
RELATIVELY COMPLEMENTED,
they are therefore
BOTH MATROID LATTICES (p. 191).

The reader should recall that
whenever we have
a set of NECESSARY and SUFFICIENT conditions
for a certain result to follow from them,
one should always realize that
it is "a" and not "the" set of
necessary and sufficient conditions
for the conclusion in question.
Thus, to draw the conclusion that
a given quadrilateral is a PARALLELOGRAM,
one set of necessary and sufficient conditions
is:
that one pair of sides be equal and parallel;
an alternative set is:
that both pairs be parallel;
and there are many other
alternative sets.
Similarly,
there are alternative ways * of stating
necessary and sufficient conditions
for a Lattice to be

(a) a MODULAR Lattice or
(b) a SEMI-MODULAR or Birkhoff Lattice,

and these alternative ways are
sometimes easier to use:

---

* See G. Birkhoff: "Lattice Theory" pp. 68 and 100.

thus,
(a) for MODULARITY,
instead of L5,
we have the EQUIVALENT demands
(in any lattice in which
all bounded chains are finite):

(1) that the Jordan-Dedekind condition
hold (p. 184)
AND
(2) that

$$d\left[x \frown y\right] + d\left[x \smile y\right] = d\left[x\right] + d\left[y\right]; \; *$$

whereas
(b) for SEMI-modularity,
we may use,
instead of the definition on p. 186,
the EQUIVALENT demands:

(1) that the J-D condition hold

AND

(2) $$d\left[x \frown y\right] + d\left[x \smile y\right] \leqq d\left[x\right] + d\left[y\right]$$

You can now easily see
by comparing (2) in (a) and in (b) above,
that MODULARITY implies
SEMI-modularity,
but NOT conversely! (see p. 188)
Thus if $A = B$
it is of course true that

$$A \leqq B,$$

whereas,
if $A \leqq B$,

it does NOT follow that
A MUST EQUAL B!

---

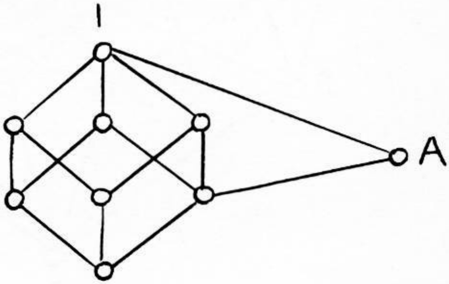
* See the definition of $d\left[x\right]$ on p. 184.

Further,
in a SEMI-MODULAR lattice of
finite length,
the following are EQUIVALENT: *

(a) It is COMPLEMENTED
IF and ONLY IF
1 (the greatest element) is
the "join" (i.e. the $\smile$) of "points";

(b) it is RELATIVELY complemented
IF and ONLY IF
EVERY element X is
the join of points.

Thus, the following Lattice



though it is COMPLEMENTED
(since 1 is the join of points),
it is NOT RELATIVELY complemented
(since A is not the join of points) —
so that
it is quite easy to see
by this method
that this is NOT a MATROID Lattice (p. 191),
and therefore CANNOT be
the lattice of
either an affine or a projective geometry. **

---

* G. Birkhoff: "Lattice Theory", p. 105.

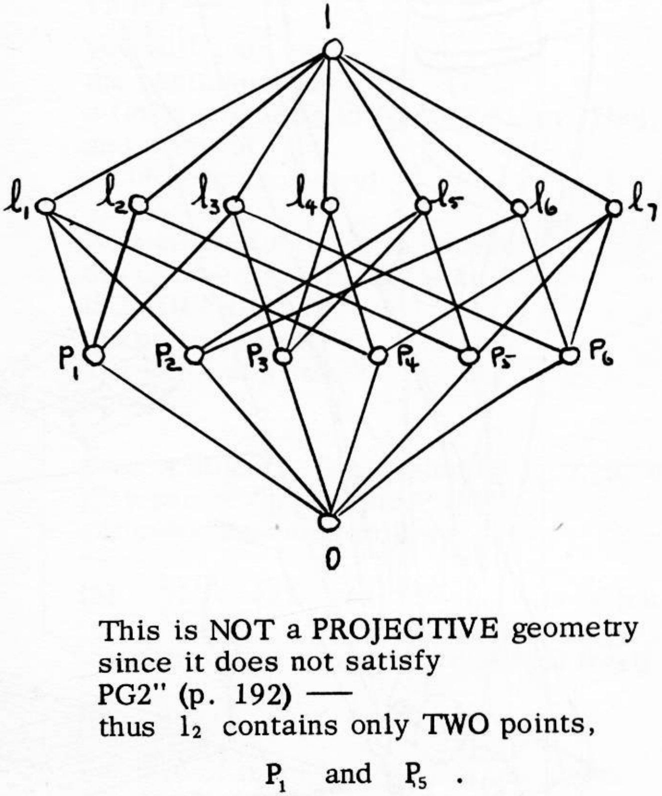
** See p. 198.

So you see that
these EQUIVALENT criteria
are very easy to use here and
come in very handy.

Furthermore, you will soon see that
this Lattice cannot represent
any geometry whatsoever,
for we shall consider in the next chapter
geometries other than
affine or projective,
and give the postulates for
a geometry in GENERAL!

XXIII. Other Geometries.

A. The reader may be interested to see geometric* lattices which are NEITHER affine NOR projective. For instance:



This is NOT a PROJECTIVE geometry since it does not satisfy PG2" (p. 192) — thus l_2 contains only TWO points,

P_1 and P_5 .

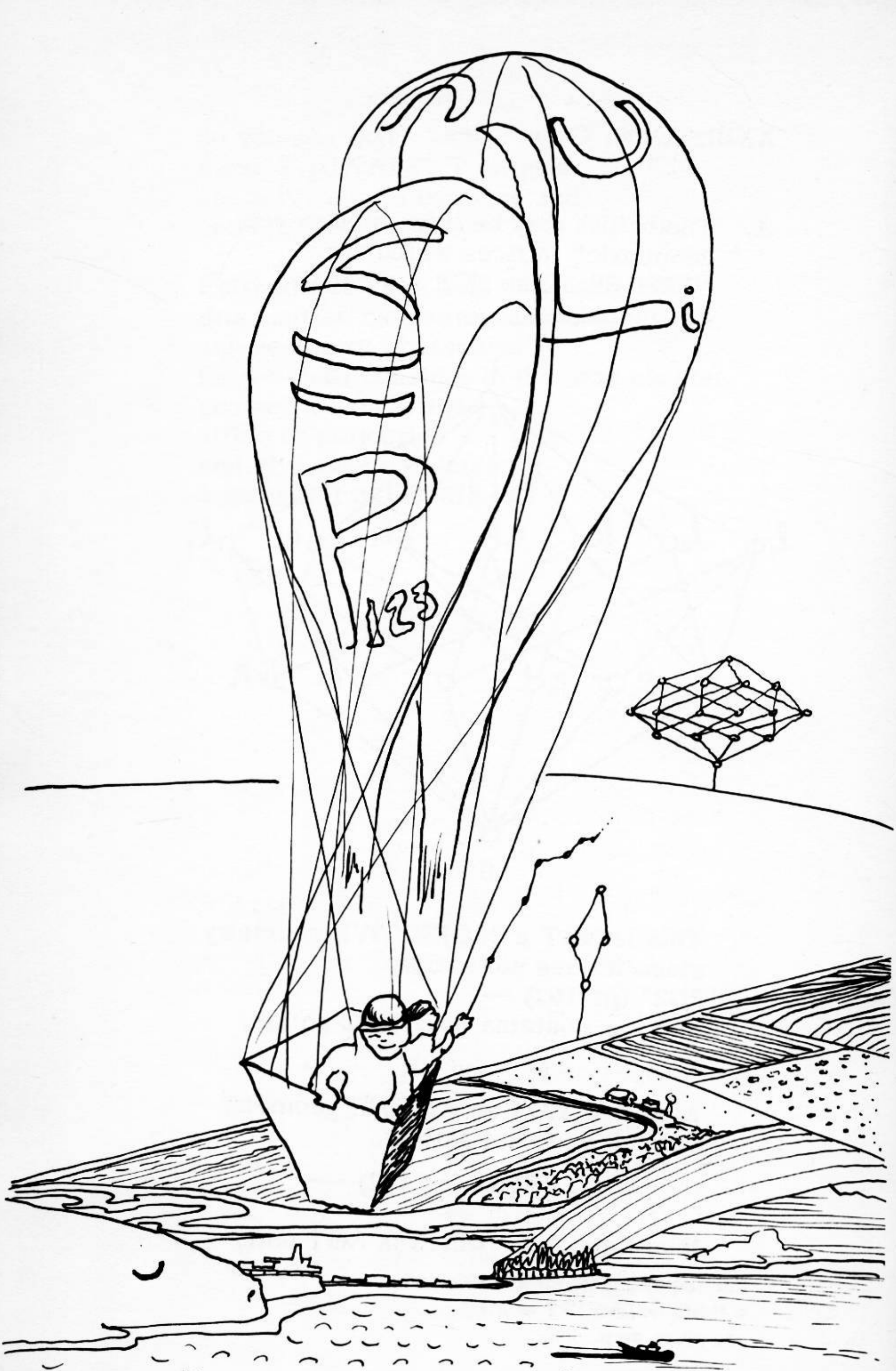
Also it is NOT an AFFINE geometry since it does not satisfy

APG2 (p. 192) —

thus through P_1 which is NOT on l_7

---

* The requirements for a lattice to be a finite GEOMETRY at all are given on p. 206.


P
123
i

there is NO line parallel to l_7,
since through P_1 there are
only three lines,
l_1, l_2, and l_3, and
l_7 INTERSECTS all of them at
P_4, P_5, and P_6, respectively.

But is it a geometry at all?
IT IS! —

you will soon see
the postulates for
a finite geometry in GENERAL (p. 206);
and even for
an infinite geometry!

It is interesting also to note that
the Geometry on p. 202 is
ISOMORPHIC with
the partitions of
a set of 4 objects, *
thus:

Take 4 objects, designated by 1, 2, 3, 4;
they can be "partitioned" in
various classes as follows:

(a) $\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ in which

each is in a separate class by itself;

(b) $\{12\}$ $\{3\}$ $\{4\}$ in which

two of them are in the same class,
but the other two are still
in separate classes.

---

* See M. Lesieur: "Des Treillis Géométriques", pp. 259ff., in the charming book referred to on p. 221.
See also p. 92 in Hans Hermes: "Einführung in die Verbandstheorie", another delightful book on Lattice Theory,
also referred to on p. 221.

Here we can have also

$\{13\} \quad \{2\} \quad \{4\}$

etc.,
and obviously there are
SIX partitions of this type;

(c) $\{123\} \quad \{4\}$, etc.

of which there are FOUR,
are there not?

(d) $\{12\} \quad \{34\}$, etc.,

of which there are THREE;
and

(e) $\{1234\}$ of which there is only ONE.

If we now order these partitions
by inclusion,
we get a Lattice which is
ISOMORPHIC with the one shown on p. 202
by setting up the
following correspondence:

(a) with the null-class, 0;
the six partitions (b) of the type

$\{12\} \quad \{3\} \quad \{4\}$

with the P_i (i = from 1 to 6);
the seven partitions (c) and (d)
with the $1_{j's}$ (j = from 1 to 7);

and (e) with the greatest element, 1.

One must of course be careful
that the correspondences are
so assigned that
the "inclusions" are valid,
thus:

if $P_1 \longleftrightarrow \{12\} \quad \{3\} \quad \{4\}$

and if

$l_1 \longleftrightarrow \{123\} \quad \{4\}$,

then we must have

$P_2 \longleftrightarrow \{13\} \quad \{2\} \quad \{4\}$ and

$P_4 \longleftrightarrow \{23\} \quad \{1\} \quad \{4\}$ since

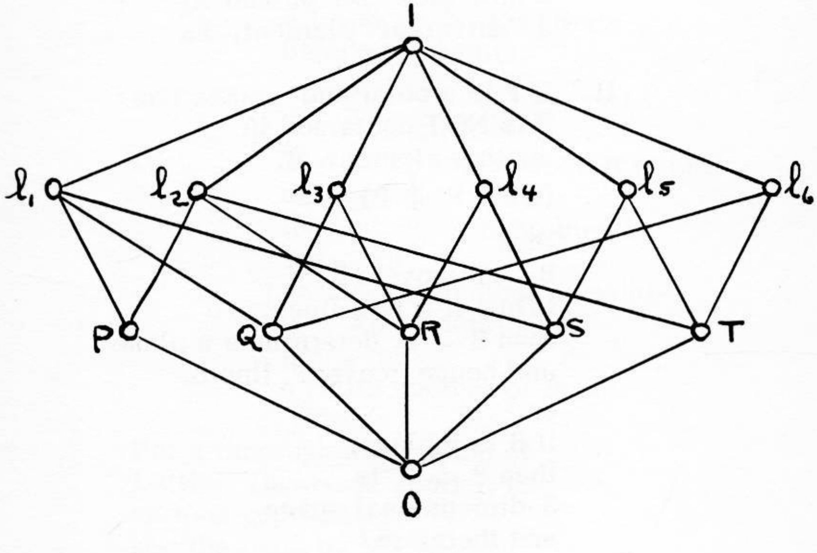
these are the three P's
which are joined to l_1 by
rising lines in the diagram,
and of course

$\{12\}$, $\{13\}$, and $\{23\}$ are

included in $\{123\}$ *

so that the partitions mentioned
correspond to the facts of
the diagram.

B. And now still another
Geometry:



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* Note that in every case,
each number NOT referred to
is understood to be in a class by itself.

Here there are five **POINTS**,
P, Q, R, S, T
and the **LINES** are:

$l_1 = P \smile Q = Q \smile S = S \smile P$
$l_2 = P \smile R = R \smile T = T \smile P$
$l_3 = Q \smile R$
$l_4 = R \smile S$
$l_5 = S \smile T$
$l_6 = T \smile Q$

Furthermore,
many other geometric lattices are
POSSIBLE because
the only requirements for
a lattice to be
a FINITE GEOMETRY at all
are the
following three simple postulates:

I. It must have
a null-element, 0, and
a "universal" element, 1.

II. If P is a point and
P is NOT contained in
another element, B,
(i.e. $P \nleq B$),

then
$B \smile P$ covers B.
(Thus if B is a line,
then $B \smile P$ determines a plane
and hence "covers" line B;

or
if B is a plane,
then $B \smile P$ is
3-dimensional space
and therefore
"covers" plane B, etc.)

III. Every element A ($\neq 0$)
is the $\smile$ (or "join" or "union")

of a finite number of points.
(See p. 200).

The characteristic properties of
such a lattice
are:
it is SEMI-MODULAR,
of FINITE LENGTH, and
RELATIVELY COMPLEMENTED.

Similarly it is quite easy also
to characterize
a still more general geometry,
whether FINITE or INFINITE! —
namely —

by retaining
postulates I. and II. on p. 206,
changing III. to

III$_1$: Every element A ($\neq 0$)
is the $\smile$ of
a finite or INFINITE
number of points.

and adding

IV: The $\smile$ of any number of points
exists, and
if point P is contained in
the $\smile$ of
an infinite number of points,
it is also contained in
the $\smile$ of
a finite number of these.

For a thorough discussion of
Lattice Theory as applied to
various geometries,
see the book by M Lesieur
referred to on p. 203 —
you will enjoy it immensely.
He gives, on p. 316,
an excellent summary of

the postulates for
the lattices of
various geometries
together with
their characteristic properties.

XXIV. Algebra of Algebras!

And so we have seen that
Lattice Theory
throws light on the study of
Boolean Algebras (Chapters XVII. and XVIII.),
Group Theory (Chapter XIX.),
Geometries (Chapters XX. - XXIII.).
Furthermore,
as Garrett Birkhoff says
in the Preface of his wonderful book,
"Lattice Theory" (1948):

> "A series of articles
> published in 1933 – 7
> by myself, von Neumann,
> Ore, Stone, and Kantorovitch
> showed the generalizations of
> Boolean Algebra to
> suitable "lattices"
> had FUNDAMENTAL APPLICATIONS to
> MODERN ALGEBRA,
> PROJECTIVE GEOMETRY,
> POINT-SET THEORY, and
> FUNCTIONAL ANALYSIS,
> as well as to
> LOGIC and
> PROBABILITY.
> Some of this work had been
> anticipated in
> two little-noticed papers by
> Dedekind,
> and the writings of
> Menger, Tarski and Fritz Klein."
> And later he adds:
> "Like its elder sister,
> group theory,
> lattice theory is a fruitful source of
> abstract concepts
> COMMON to

TRADITIONALLY UNRELATED
branches of mathematics.
Both subjects are based on
postulates of an
EXTREMELY SIMPLE and
GENERAL nature.
It was this which
convinced me from the first
that lattice theory
was destined to play —
indeed, already did play
implicitly —
a FUNDAMENTAL ROLE
in mathematics." *

And we have seen that
the various properties of lattices
that have helped in
making some of these applications
have been very interesting
in themselves,
and certainly would NOT be obvious
from a casual glance at
a few diagrams!

I cannot leave this
marvellous domain
without making at least
a minimum reference to
the fact that
we can now operate
with systems in which
the ELEMENTS are themselves
WHOLE SYSTEMS!

---

* I took the liberty of writing this
quotation
in the short-line form of
the rest of the book,
and of using capitals,
in the hope that
this will serve to EMPHASIZE
the ideas for
the layman.

We can "add", "multiply", etc.,
whole posets, lattices, etc. —
and it is therefore
natural to ask whether
the SUM or PRODUCT of two POSETS
is itself a POSET or not,
whether the SUM or PRODUCT of
two LATTICES
is itself a LATTICE or not,
and what are the other laws
governing the operations here.

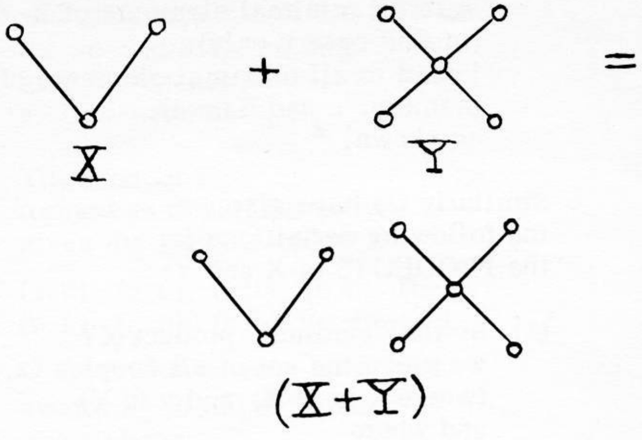
First we define
SUM and PRODUCT of POSETS,
as follows:

(1) The "cardinal" sum of
two posets, X and Y,
designated by

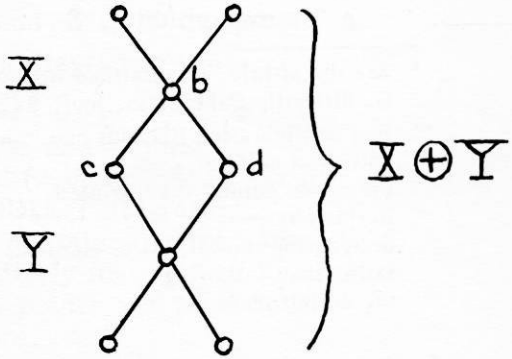
$$X + Y$$

is defined thus:
it is the set of all elements
in X or Y,
where $\leqq$ KEEPS its meaning
WITHIN X and Y,
but where
$x \leqq y$ (x is in X and y in Y)
NEVER holds,
and X and Y are considered as
disjoint
(i.e. they have NO element in common).

From the viewpoint of
their diagrams,
this means that
"cardinal" addition of two posets
is accomplished by
putting their diagrams
next to each other,
side by side,
thus:



(2) The "ordinal" sum of
two non-overlapping posets,
X and Y,
designated by $X \oplus Y$,
is the set of all elements
in X and Y,
where $\leqq$ KEEPS its meaning
WITHIN X and Y,
but unlike (1),
$y \leqq x$ for ALL
x's in X and y's in Y —
in other words,
their diagrams are placed
so that
X is ABOVE Y,
thus



with all minimal elements of X
(in this case b only)
joined to all maximal elements of Y
(namely, c and d here),
as shown. *

Similarly we have also
the following definitions for
the PRODUCTS of X and Y:

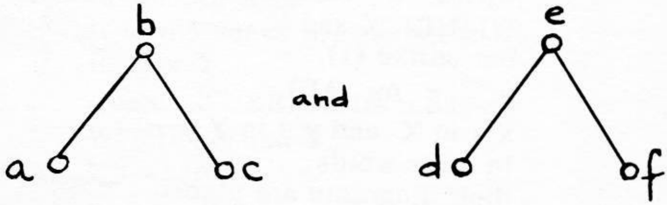
(1) By the "cardinal" product XY,
we mean the set of all couples (x, y)
(where x is in X, and y in Y)
and where

$(x, y) \leqq (x', y')$ means that

$x \leqq x'$ in X and $y \leqq y'$ in Y.

Let me illustrate this
with diagrams:

Consider the posets



the first of these has
two "points", a and c, **
both "contained" in the "line" b,
and therefore represents
a "line-segment", S_1;

---

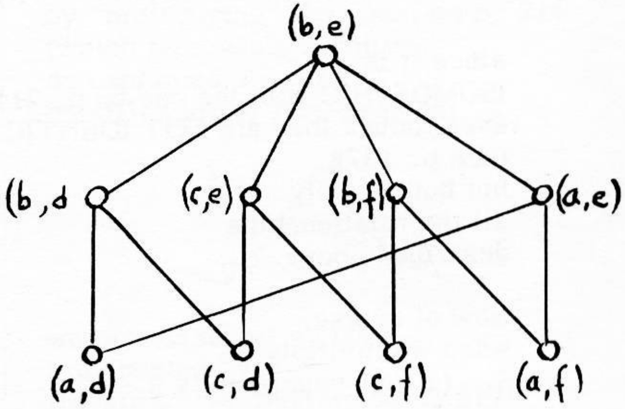
* See the article "Generalized Arithmetic", by
G. Birkhoff, Duke Math. Jour. 9 (1942) pp. 283-302.

** In a configuration like this one,
without a null-set,
the word "points" may be used
in this way —
designating the MINIMAL elements —
rather than following
the definition on p. 103.

and similarly
the second diagram here
also represents
a "line-segment", S_1 .

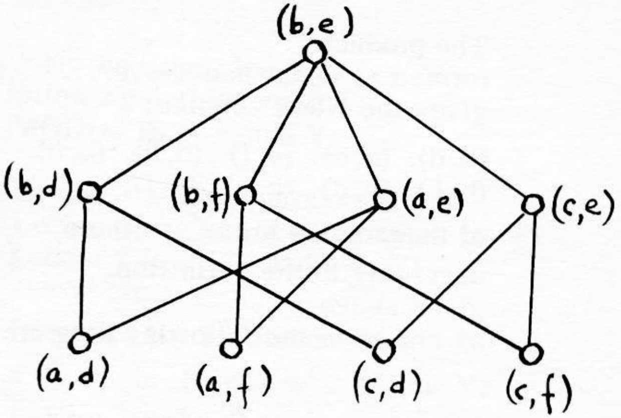
The product,
formed as described above,
gives the NINE couples:

(a, d), (a, e), (a, f), (b, d), (b, e),
(b, f), (c, d), (c, e), (c, f);
of these some are $\leqq$ others,
according to the definition
given above,
as shown in the following diagram:



where (a, d) $\leqq$ (b, d) because
a $\leqq$ b and d $\leqq$ d in the
two given diagrams;
also (c, d) $\leqq$ (b, d);
so that the two "points"
(a, d) and (c, d) in
this PRODUCT diagram
are both contained in the "line" (b, d).
And similarly for
the other points and lines in
this figure.

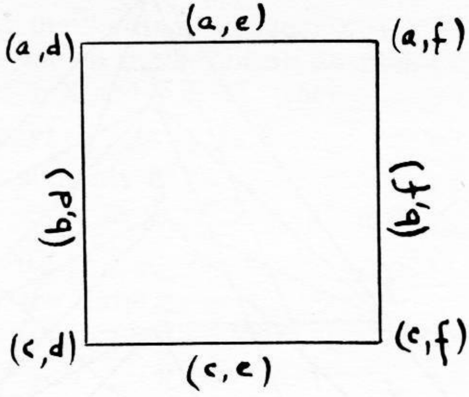
Note that
one might have drawn
the following diagram instead:



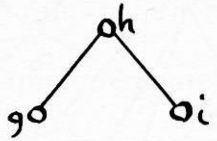
since it is
ISOMORPHIC with the one on p. 214,
even though they are NOT IDENTICAL
(see p. 137),
but both satisfy
all the relationships
described above.

Now of course,
when we multiply
S_1 (a line-segment) by S_1

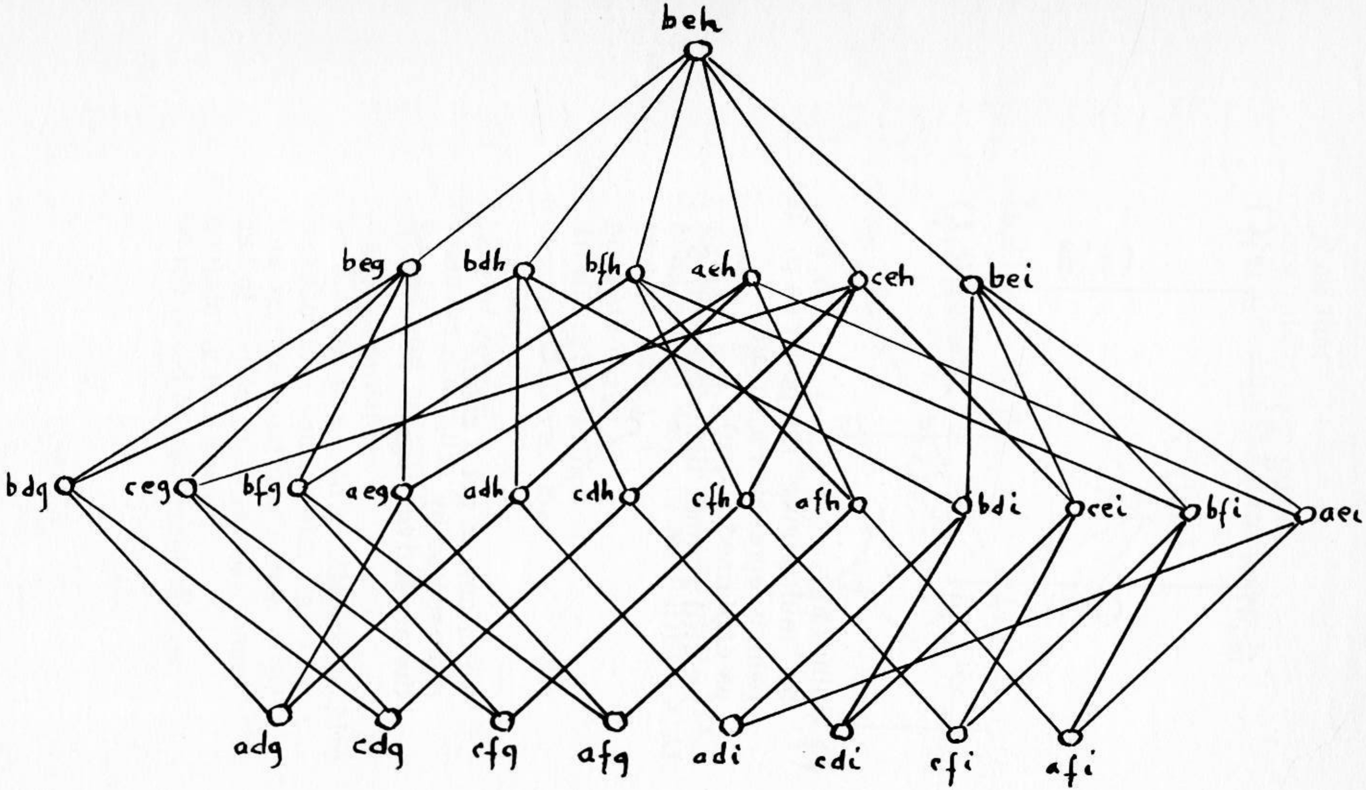
we get a square,
so that the poset on p. 214 (or p. 215)
represents a square with its
four vertices and four sides,
and of course (b, e) is the "plane"
containing this square
which could be represented thus:


(a,e)
(a,d)
(a,f)
(b,d)
(b,f)
(c,d)
(c,f)
(c,e)

Similarly
by "multiplying" the poset on p. 214
(which represents a square,
as explained above)
by still another line-segment, S_1 :



we of course get a cube,
represented by
the attractive diagram on p. 217!


beh
beg
bdh
bfh
aeh
ceh
bei
bdg
ceg
bfg
aeg
adh
cdh
cfh
afh
bdi
cei
bfi
aei
adg
cdg
cfg
afg
adi
cdi
cfi
afi

POSET representing a CUBE.

And so on.

(2) By the "ordinal" product $X \circ Y$
is meant the set of all couples (x, y)
(x in X and y in Y), where

$(x, y) \leqq (x', y')$

if and only if

$x < x'$ in X or $x = x'$ in X

AND $y \leqq y'$ in Y.

?

In the finite case,
we can construct the diagram of

$$X \circ Y$$

from the diagrams of X and Y
as follows:

In each circle representing
an element x of X,
put a replica Y_x of Y.

Then draw segments
from all the minimal elements of
each Y_x and
all the maximal elements of
each $Y_{x'}$
for x' covering x.

Let us now see
some of the postulates for
these operations:

I. CARDINAL operations:

(a) commutative laws:

$$X + Y = Y + X$$
$$XY = YX$$

(b) associative laws:

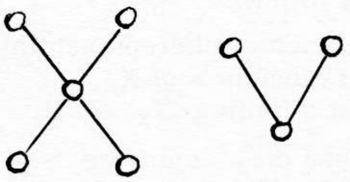
$$X + (Y + Z) = (X + Y) + Z$$
$$X(YZ) = (XY)Z$$

(c) distributive law:

$$X\,(Y + Z) = XY + XZ$$
$$(X + Y)\,Z = XZ + YZ$$

and others.

Note that the = sign here
means "is isomorphic with".
Thus,
in the diagram on p. 212 for
the cardinal sum $X + Y$,
it is easy to see that
it is isomorphic with



which represents the sum $Y + X$.
And so on.

II. ORDINAL operations:

(a) Note that in both
ordinal ADDITION and
ordinal MULTIPLICATION,
commutativity is
not demanded, since
it does not always hold.*

(b) the associative laws do hold:

$$X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$$
$$X \circ (Y \circ Z) = (X \circ Y) \circ Z$$

(c) the distributive law:

$$(A \oplus B) \circ C = A \circ C \oplus B \circ C.$$

---

* See G. Birkhoff: "Lattice Theory", p. 9.

For other interesting relationships here,
see the paper by G. Birkhoff
referred to on p. 213.

And similarly for
cardinal and ordinal operations
with LATTICES, etc.

Even with this
meager discussion of
these ultra modern systems,
the reader can no doubt sense
the power of
these wonderful tools of thought.

I need hardly say that
this little book is
the merest beginning of
an introduction to
this beautiful subject of
Lattice Theory
and I take pleasure in
referring the interested reader to
the following three
fascinating books:

(1) Garrett Birkhoff's
"Lattice Theory"
(the 1948 edition),
published by the
American Mathematical Society
as Volume 25 of their
Colloquium Publications.
This book has everything!
It is thorough, complete,
and shows the hand of
a CREATIVE mathematician,
and not the dismal approach of
some depressing "text-books".
The other two books
though one is in French,
and the other in German,
are so charmingly written
and have so many

delightful illustrations,
that I hope you will have
as much fun with them
as I have had.
They were recommended to me
by Garrett Birkhoff himself.
They are:

(2) Leçons sur la
1. Théorie des Treillis *
2. Des Structures Algébriques Ordonnées
3. Des Treillis Géométriques

par

1. R. Croisot;
2. Mme. M.L. Dubreil-Jacotin;
3. L. Lesieur;

and

(3) Einführung in die
Verbandstheorie *
von
Hans Hermes.

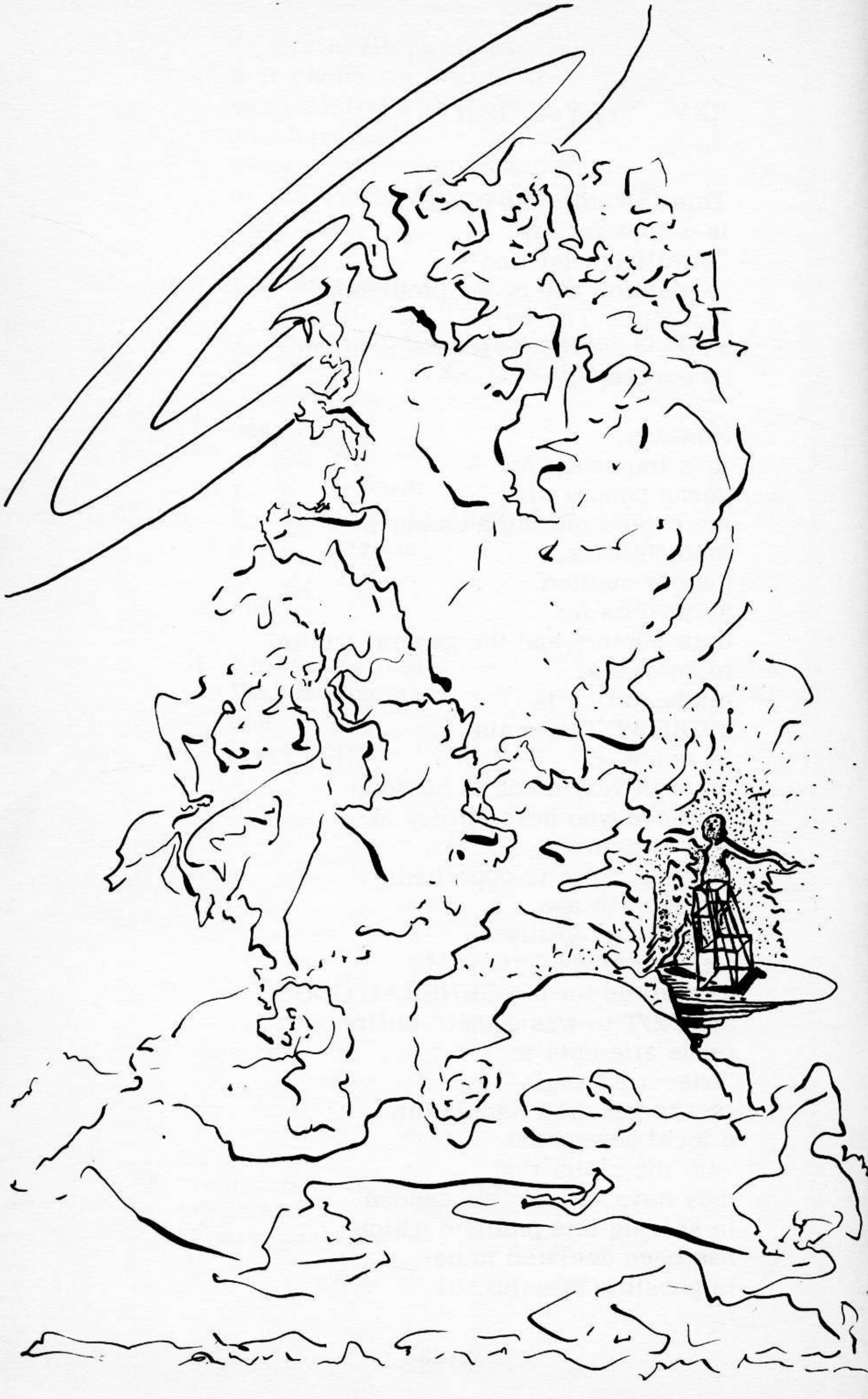
---

* "Treillis" is the French word for
LATTICE and
"Verband" is the German word for it.

XXV. Try Your Skill.

This is hardly the place,
in a book for the
"intelligent layman",
to mention any of the problems
in Lattice Theory
which have NOT YET been solved
by experts!

However,
it is important for
young people who
are contemplating a career in
mathematics,
pure or applied,
as well as for
their parents and the general public,
to know that
mathematics is
a GROWING domain
in which
there is very much to be done
by those who have ability in
this direction,
and that there is opportunity
for them to use
this native ability
for their OWN GROWTH
as well as for the GENERAL GOOD —
and NOT to waste their ability on
futile attempts to
"trisect an angle"
just to get their names into
a local newspaper
with the claim that
they have at last "succeeded"
in solving this problem which
has been declared to be
impossible of solution!



For youth always wants to do
the "impossible",
and thus
"go their elders one better".

Well and good!
BUT
the trisection problem
has already been solved!
It has been PROVED
by Group Theory,
that the general angle
CANNOT be trisected by means of
straight edge and compasses alone
(this is what the trisection problem
demanded).
It would therefore be
a much better use of
his time and strength for
a young student
to learn Group Theory so as to
understand this PROOF
rather than for him
to try again,
with his inadequate knowledge,
to struggle against something
which is already known.

And so,
in this spirit,
let us refer,
at the end of this chapter,
to some REALLY UNSOLVED problems in
Lattice Theory,
in which a youngster may hope for
SUCCESS,
provided that he equips himself with
the necessary modern tools,
so that he can
get his teeth into something
and take
a good, nourishing, tasty bite!

But first
the reader may be interested in
trying his skill on
a few "exercises" which
are NOT unsolved problems,
but rather results that have been
already obtained and which
are put in here to
give the reader a little experience in
manipulating the ideas
presented in this book,
so he can have some fun with them!

A FEW EXERCISES.

(1) Show that for any three integers,
a, b, c:

$a L (b G c) = (a L b) G (a L c)$

where xLy means
"the least common multiple of x and y",
and xGy means
"the greatest common divisor of x and y."

Thus for $a = 2$, $b = 3$, $c = 5$:

$2L(3G5) = 2L1 = 2$

and

$(2L3) G (2L5) = 6G10 = 2$;

and therefore

$2L(3G5) = (2L3) G (2L5)$.

That is,
the operation L is
Distributive over the operation G.
(See pp. 43, 44).

(2) Show that every chain is
a Distributive Lattice and
therefore also Modular.

Show that the lattice of

(3) a. ALL the subgroups of A_4 is
NOT modular;

b. the INVARIANT (NORMAL)
subgroups of A_4 IS modular.
(See pp. 174ff).

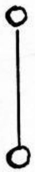
(4) Why is any sublattice of
a modular lattice
itself modular?

(5) Show that,
if a lattice is NOT Distributive
then it has a sublattice of
order 5, whose diagram is
isomorphic with
EITHER of the two diagrams on
p. 153.

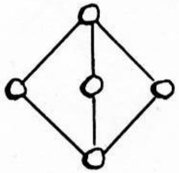
(6) Show that if a lattice is
NOT Modular,
then it contains a sublattice whose
diagram is isomorphic with
the one on p. 149.

(7) Show that
the "partitions" (p. 203) of
1, 2, and 3 objects are given by

Z_1



Z_2



Z_3

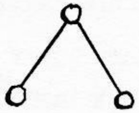
respectively.
Note: Z stands for "Zerlegung",
the German word for "partition".
(See p. 92 in the book by
Hans Hermes, referred to on p. 221).

You already know that Z_3 is
Modular (p. 148).
Show that Z_4 (p. 203) is
NON-Modular,
by taking the subset of 5 elements:

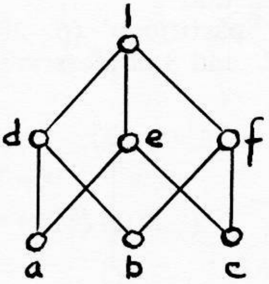
0, (12), (12) (34), (13) (24), (1234)

and showing that this subset is isomorphic with
the non-modular lattice on p. 149.

(8) Taking the "line-segment" *



and the "triangle" **



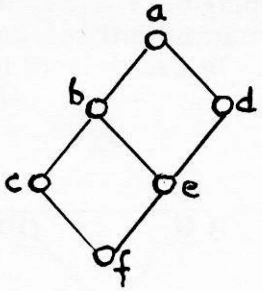
find their cardinal product *
thereby obtaining the diagram for
"a triangular prism".

---

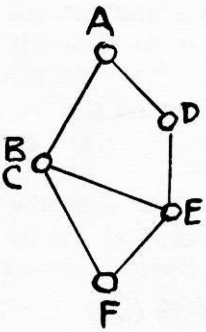
* See pp. 213ff.

** This has 3 "points" a, b, c, which determine 3 "lines", d, e, f, in a "plane", 1,
and therefore represents a triangle, does it not?

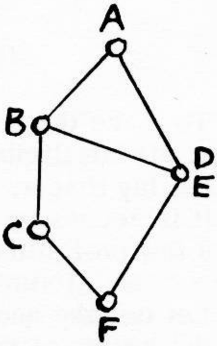
(9) Show that the lattice:



is Distributive *
(and therefore modular)
and that it has
a $\smile$ -homomorphic image **
as well as
a $\frown$ -homomorphic image **
which are both
NON-modular
(and therefore certainly
NOT distributive),
namely



and



respectively. †

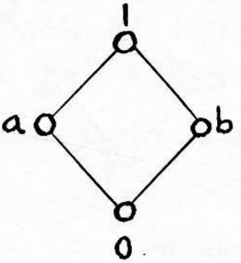
---

* p. 152.

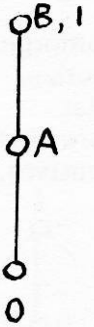
** p. 142.

† See Theorie des Treillis by
M. Croisot,
p. 70, ex. 1 and p. 81, ex. 2.

(10) Show that the following is a
join-homomorphic ($\cup$ -homomorphic)
mapping of
the matroid lattice:



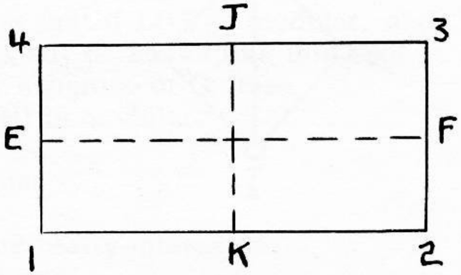
onto the chain



To make the next few exercises
at all intelligible to
the lay reader,
it is necessary to make
a few preliminary remarks:

Let us take another look at
the lattice of the
Alternating Group, A_4, on p. 174:
there the element h
represents the subgroup of A_4
consisting of the four elements
I, (12)(34), (13)(24), (14)(23).

This subgroup is often referred to as
the "Four-Group"
(so named by Felix Klein)
and is associated with
the Group of Symmetries of
the rectangle:



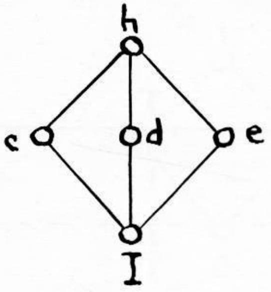
in which the elements are
the "flops" referred to on p. 16,
each of which leaves
the rectangle unchanged
although the vertices are
changed around,
thus:
the element (12)(34)
flops the rectangle over
along the axis of symmetry JK,
thus putting vertex 1
where vertex 2 now is,
2 where 1 is,
4 where 3 is,
and 3 where 4 is,
all of which is
briefly summarized in the symbol

(12) (34).

Similarly
(14) (23) represents
the "flop" along the
axis of symmetry EF,
and finally
the element (13) (24),
represents the

"double-flop" consisting of
BOTH the previous flops.

The lattice of this Group is



which is a SUBlattice (see pp. 121ff) of
the lattice of A_4 on p. 174,
and which
you will recognize to be
the NON-DISTRIBUTIVE, MODULAR lattice
of p. 161.

In the light of the above discussion of
the FOUR-GROUP,
let me mention a few
"exercises" from Birkhoff's
"Lattice Theory"
which may interest you.
But let me say again that these
"exercises" are
NOT UNSOLVED problems,
but rather results that have been
already obtained and
are given in his book as
"exercises"
for you to flex your "muscles" on:

(11) Birkhoff, p. 98, ex. 1:

(a) Determine L(G) * if

---

* See p. 175.

G is the dihedral group of
symmetries of a regular pentagon,
and show that it is modular.

(b) Same problem for the
regular 15-gon,
but show that L(G) is NON-modular.

(12) Birkhoff, p. 98, ex. 2(a):
Show that if L(G) is modular, and
H is any homomorphic image of
any subgroup of G, then
L (H) is modular.

And so on.

If you are really interested
you will find
many fascinating exercises in
all three books mentioned on
pages 220 and 221.

Finally,
without going into
any further explanations,
may I refer to the following
from Birkhoff's book
merely to
whet the reader's appetite for more:

(a) "The logic of quantum mechanics is
an ortho-complemented
modular lattice." *

(b) "A noteworthy algebra of logic
has been developed in which
the identity $(x')' = x$ is denied,
and replaced by the weaker

INEQUALITY $(x')' \leqq x$.

That is,

---

* G. Birkhoff: "Lattice Theory", p. 193.

x implies (x')' is admitted, but
(x')' implies x is denied.
This is Heyting's formulation of
intuitionist or Brouwerian logic." *

The following lattice-theoretic
interpretation of this
is due to Birkhoff (p. 197):

"A Brouwerian logic is
the dual of
a relatively pseudo-complemented
lattice."

(13) An interesting exercise
in this connection is given by him
on p. 196, ex. 6(a):
Show that
any finite distributive lattice
is a Brouwerian logic.

And, in anticipation of
the UNSOLVED problems to come,
may I give one more quotation
from Birkhoff (p. 197),
which may stir
the creative powers of
some readers:
"Interesting algebraic systems,
describing PROPOSITIONAL CALCULI
with simply ordered sets of modes,
have been proposed by
Lukasiewicz and Tarski,
and by Post . . . .
The author can see no valid reason for
this emphasis on simple ordering.
It would seem worthwhile to construct
PROPOSITIONAL CALCULI based on
non-distributive lattices of
truth-values —
say, on the two
non-distributive lattices of 5 elements.**

---

* G. Birkhoff: "Lattice Theory", p. 195.

** See p. 153.

In my own attempt to do this,
I have been troubled by
the problem as to
how the truth-values of p and q
should determine
the truth-values of $p \longrightarrow q$ and p'."

Of course the layman
can get only the vaguest idea
of what this is all about,
and can hardly be expected to
SOLVE the PROBLEMS which follow.
But
if all this
inspires the YOUNG reader
to study the subject further,
this little book will have
served its purpose.
And if, further,
it inspires ALL readers to
DO SOMETHING about
Part III. of this book,
we may even have
a BETTER WORLD!

Some UNSOLVED PROBLEMS.

And now
if you are curious to see
a few samples of
problems which have
NOT YET been solved
by anyone,
here are a few:

(1) Birkhoff, p. 98, Problem 36:
Find conditions on G which are
necessary and sufficient for
L(G) to be semi-modular.
For its dual to be semi-modular.

(2) Ditto, Problem 37:
Find necessary and sufficient
conditions on G

in order that
L(G) be self-dual.

(3) Ditto, Problem 38:
Find necessary and sufficient
conditions on G
in order that
L(G) be complemented.

And so on.

Birkhoff lists
ONE HUNDRED AND ELEVEN
such UNSOLVED PROBLEMS!

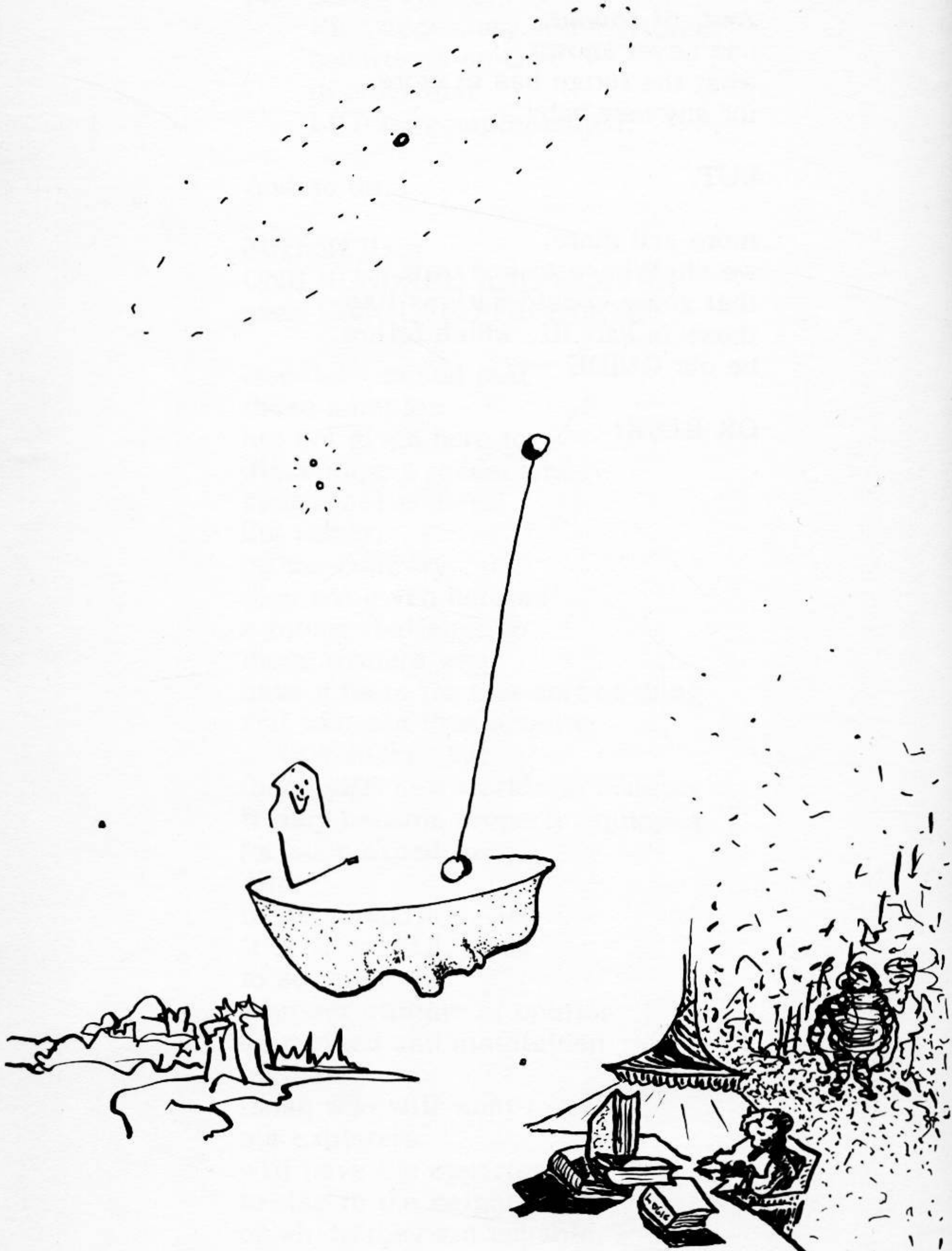
Needless to add that
these samples
are not given here to
discourage a reader who
cannot solve them!
But rather,
on the contrary,
they are given here as
a future challenge to
those readers who
have a taste for this sort of thing
and who can thus acquire
a NEW HOPE that
there ARE new worlds to conquer
if they become properly equipped
for such expeditions.
And,
in the meantime,
it is up to ALL of us
to see to it that
a proper climate of opinion
be created and maintained
so that
those who will want to be
our explorers
will have the opportunity
to rise to the heights
of which they are capable,

to the greater glory of mankind,
as well as to
their own deep satisfaction.
And, of course,
one never knows
what the future has in store
for any new baby.

BUT,

more and more,
we shall have to see to it
that some considerations like
those in Part III. which follow,
be our GUIDE —

OR ELSE!


LOGIC

PART III.

BEHIND THE SCENES.

XXVI. METAmathematics.

Some readers may be surprised that
I have not mentioned
the very important
Duality Principle,
which is so helpful in
the study of Lattice Theory,
as well as in
other branches of mathematics.
The fact is that
this Principle is NOT
a mathematical principle at all
but is ABOUT mathematics,
and is therefore
OUTSIDE of mathematics,
belonging to what is known as
METAMATHEMATICS.

Let me first say a few words,
in general,
about this most important subject,
before discussing
the Duality Principle in particular.

For example,
suppose you have
a set of axioms for some system,
it is of BASIC IMPORTANCE that

(1) these axioms
MUST NOT CONTRADICT EACH OTHER!

And it is sometimes desirable that
(2) the axioms be
INDEPENDENT of each other, and
(3) they be sufficient to
DETERMINE

a particular branch of
mathematics —
like Euclidean Geometry,
for instance.
In this case
the postulate set is said to be
COMPLETE.
May I warn the reader here that
these concepts
(consistency, independence, completeness),
though they may seem fairly simple
are indeed very SUBTLE! *

A discussion of such considerations,
though of the utmost importance,
is NOT WITHIN
ANY branch of mathematics.
Thus,
if you wish to PROVE that
a certain postulate is
INDEPENDENT of
the other postulates of a system,
this cannot be done in
the usual sense of "proof",
i.e. deriving a result from
the given postulates of the system,
as in proving a theorem
within the system.
The independence of a postulate
is "proved" in the following manner:

Suppose you have a system having
FIVE postulates —
call them a, b, c, d, e, —
and you wish to show that
e is independent of the other four.
IF you can exhibit a NEW system
in which the postulates a, b, c, d
are retained unchanged,
and e is replaced by its

---

* See Appendix III., p. 270.

CONTRADICTORY,
call it e', —
that is,
a new system having the postulates
a, b, c, d, e',
then
you can see that
e is independent of
a, b, c, and d,
because —
if e were derivable from
a, b, c, d,
then
the system a, b, c, d, e'
would be a SELF-contradictory system
(containing the
contradictory statements
e and e',
since we assumed that
e FOLLOWS from a, b, c, d).
But
if you were able to
actually produce a model of
the NEW system,
thus showing that
it is a POSSIBLE system
and hence NOT a SELF-contradictory one —
it would show that
e is not somewhere hidden within it,
and therefore
e cannot follow from a, b, c, d
and is therefore
INDEPENDENT of them.
And similarly
to show that
a, b, c, and d are also
each independent of the others,
you would have to construct in all,
FIVE NEW SYSTEMS,
each of which has
FOUR postulates in common with
the original given system,

and one postulate which
CONTRADICTS the remaining postulate,
in turn.

This procedure would be
much clearer if
you saw it actually carried out;
and so I recommend
the charming little book
previously referred to:
"The Fundamental Propositions of
Algebra" by E.V. Huntington, *
in which the author
does this very thing for
the algebra of complex numbers,
setting up
27 new algebras,
each of which has
26 postulates in common with
the algebra of complex numbers,
and a 27th one which
contradicts the omitted one —
thus
proving the independence of
all the original 27.

Perhaps this seems to you to be
too laborious to be interesting!
Not at all —
you will be so delighted with
some of the funny little algebras
which Huntington makes up
for this purpose,
and it will throw so much light
on ordinary algebra,
in such an amusing way,
that,
I assure you,
you have never had such fun
with algebra
in your life!

---

* See the Bibliography at the end of this book.

I cannot leave this subject of
INDEPENDENCE
without at least referring to
THE Great Discovery
in the early part of
the 19th century,
in which,
after 2000 years of failure to
prove Euclid's famous
PARALLEL POSTULATE
(thinking that it was really
not a postulate at all,
and should therefore be provable)
it was finally realized that
it was INDEPENDENT of the
other postulates,
and was a bona fide postulate after all.
And indeed
this independence was proved by
the actual construction of the
NON-Euclidean geometries
which have the other postulates
in common with Euclidean Geometry
but in which
Euclid's parallel postulate
was omitted and
REPLACED by
its contradictory * —
just as Huntington did for Algebra.

Perhaps you are wondering why it took
GENERATIONS of mathematicians
2000 years
to prove the independence of
a SINGLE POSTULATE,
when Huntington
recently proved the independence of
TWENTY-SEVEN postulates
in a short time!
Well, the fact is that

---

* See "Human Values of Modern Mathematics" in this series.

in those old days
(before the 19th century)
no one realized that
it was PERMISSIBLE to
CHANGE postulates,
as discussed above,
for, to them,
postulates were
"self-evident truths"
and therefore
NOT SUBJECT TO CHANGE!
But when it was finally realized
in the early part of the 19th century,
that postulates are
MAN-MADE ASSUMPTIONS,
the psychological blockage
was removed,
and man found himself able
to CREATE new systems —
and thus
the floodgates of DISCOVERY
were opened wide!

Do you wonder that
I call the above-mentioned
CHANGE of ATTITUDE toward
the NATURE of POSTULATES,
THE Great Discovery which led to
all the others —
the construction of
the wealth of various systems,
some of which have been described
in this little book!

And finally,
let me refer you to
"Elementary Topics in
Mathematical Logic" by
Alonzo Church, *
in which,
in developing SET THEORY,
he explicitly states that

---

* See the Bibliography at the end of this book.

he will introduce
TWO KINDS of postulates:

(1) those WITHIN the system
which are entirely expressible
in the notation of set theory,
and

(2) those which are
NOT IN
but ABOUT
set theory,
which "require for their expression
the use of English* words
in addition to
the notation of set theory."

This second kind of postulates
serves the following functions:
they EITHER

(a) give permission to ASSERT
any sentence of a certain kind or form,
in which case they are called
SCHEMATA, or

(b) give permission to INFER
a sentence of a certain form
from one or more
sentences (premises) of
specified forms,
in which case they are called
RULES of INFERENCE. **

He then proceeds to
number the first kind, (1),
with ARABIC numerals,

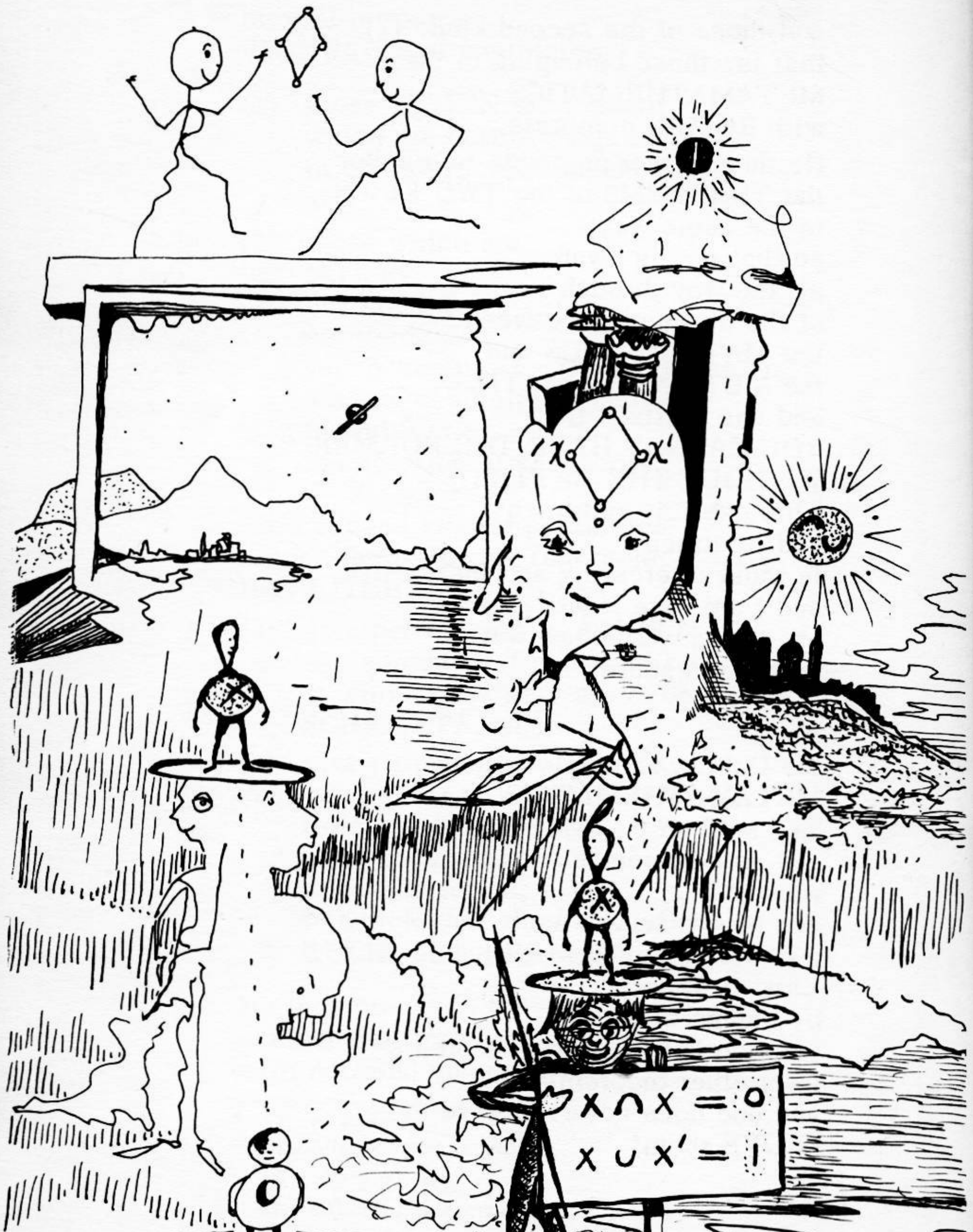
---

* Or, of course,
any other native language.

** cf., for example,
the RULES of inference for
the logic of Syllogisms
(see p. 118 of "Mits, Wits and Logic,"
Item (4) in the Bibliography, p. 286);
thus one such RULE is:
"From two negative premises
nothing can be inferred.";
etc.

and those of the second kind, **(2)**, —
that is, those belonging to the
METAMATHEMATICS —
with ROMAN numerals.
He then further proceeds to number
the THEOREMS of the TWO KINDS
in the same way,
so that one is aware
all the way through
of the distinction between
the MATHEMATICS and
the METAMATHEMATICS,
and one realizes that
YOU CANNOT HAVE THE FORMER
WITHOUT THE LATTER.
This explicit statement
and treatment
is most interesting and instructive
and I am sure it will
open up a whole new world
to many readers.

Now,
the Duality Principle belongs to
METAMATHEMATICS,
as I said on p. 237.
Let us now see
what it is,
and how it serves as
a powerful tool in
Lattice Theory,
by means of which
one can INFER many theorems
from other theorems —
and one does not even have to
PROVE them!


x
x'
x ∩ x' = 0
x ∪ x' = 1

XXVII. The Duality Principle.

This Principle is
an excellent illustration of
a principle of
METAmathematics,
as you will soon see.

Let us go back first to
P1, P2, P3, (page 76),
the three characteristic properties of
a Partially Ordered SET —
namely that it is

P1: reflexive
P2: anti-symmetric
P3: transitive.

Now it is easy to see that
these three properties
hold also for the relation $\geqq$,
since it is true that
$x \geqq x$ (reflexive)
($x \geqq y$ AND $y \geqq x$) imply $x = y$ (anti-symmetric)
($x \geqq y$ AND $y \geqq z$) imply $x \geqq z$ (transitive).

Hence
any theorem which is true in
every poset
remains true if
the symbols $\leqq$ and $\geqq$ are
interchanged throughout the
statement of the theorem.

THIS IS THE DUALITY PRINCIPLE.

Note that it shows how
from one true theorem
you can automatically get

another true theorem,
and is therefore
a principle ABOUT theorems
rather than a theorem itself.
It is therefore
a principle of METAmathematics
since it is a
RULE of INFERENCE
as described by Church (p. 243).

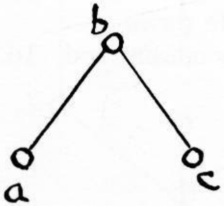
Further,
note that if
in the definition of $\frown$ in I. on p. 111,
we replace $\leqq$ by $\geqq$,
it changes the two properties
from a definition of $\frown$ to
a definition of $\smile$,
as in II. on pp. 111, 112.
Consequently
the Duality Principle
may now be applied,
not only to posets as on p. 245 ,
but also to
LATTICES if
one interchanges $\frown$ and $\smile$
throughout the statement of
a theorem in Lattice Theory,
thus obtaining
a new true theorem from
a given true theorem,
WITHOUT PROOF!

Thus, for example,
in the theorem on p. 151
in which it was PROVED that
L6' implies L6",
we can
WITHOUT FURTHER PROOF,
conclude that
L6" implies L6'
by merely interchanging $\frown$'s and $\smile$'s !!

Similarly
in L1 – L4 (p. 126),
by interchanging $\frown$ and $\smile$
the first identity of each pair
(L1, etc.)
changes to
the second of the pair,
thus
$x \frown x = x$ becomes $x \smile x = x$,
and so on for
L2, L3, and L4,
so that,
armed with the
Duality Principle,
only ONE of each pair
need be proved, *
since the OTHER of each pair
FOLLOWS automatically by
the Duality Principle.

And this applies also to any theorem
which is a consequence of L1 – L4,
that is,
for any such theorem which is
PROVED,
you get another free gratis!

Consider now the POSET

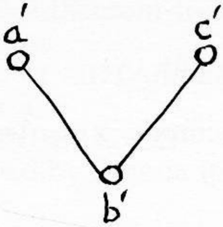


I.

---

* When, as on p. 126
they are not taken as postulates.
And when they are taken as postulates
(as on pp. 106, 107),
we need assume only one of each pair,
since the other follows by the
Duality Principle, as above.

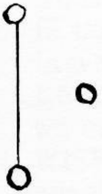
and turn it upside down
so that it becomes



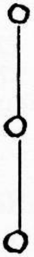
II.

with each element x of I. becoming
x' in II;
here, then
$a \leqq b$ in I. gives $a' \geqq b'$ in II.,
and similarly for
ANY pair of elements.

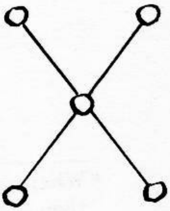
Now,
when any TWO POSETS,
like I. and II.
have $x \leqq y$ in I. and $x' \geqq y'$ in II.,
they are called
DUAL posets,
each being the DUAL of the other.
Some posets,
when turned upside down,
leave the diagram unchanged, like



(1)



(2)



(3)

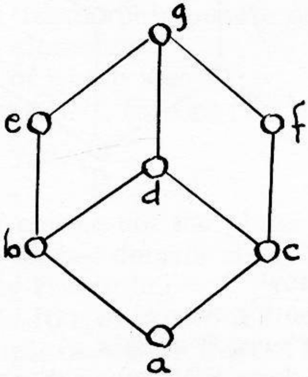
etc.

in which case
they are called
SELF-DUAL.

And, for TWO LATTICES,
if $x \frown y = z$ in one
and $x' \smile y' = z'$ in the other,
then these are
DUAL lattices.

Let me caution the reader
not to use
the Duality Principle
CARELESSLY,
by showing an interesting case:

Consider the
SEMI-MODULAR LATTICE on p. 186,
which is characterized by
the condition:
if x and y cover a, and $x \neq y$,
then $x \smile y$ covers x and y (p. 186):

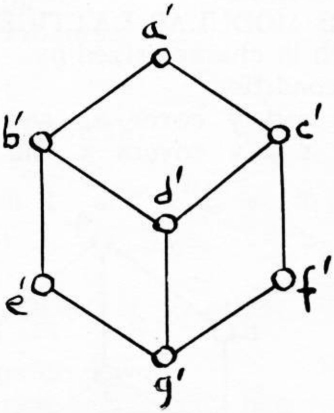


Here, for instance,
b and c both "cover" a,
and $b \smile c$, which is equal to d,
"covers" both b and c.
And similarly for

e and d which both cover b,
$e \smile d = g$ covers e and d,
etc.
Now,
it might seem that
if you turned this diagram
upside down,
the above-mentioned condition
would be preserved and
the new DUAL lattice would be also
a SEMI-MODULAR LATTICE.

BUT THIS IS NOT SO!

For, in



although now
e' and f' both cover g',
yet $e' \smile f'$ which equals a'
obviously does NOT cover e' and f',
as demanded by the condition on p. 186,
and so this lattice is
NOT semi-modular;
that is to say,
the DUAL of a SEMI-MODULAR lattice
is not necessarily itself
semi-modular.

It is interesting to note here that
the DUAL of a finite SEMI-MODULAR lattice L
IS semi-modular IF and ONLY IF
L is modular!

The reader may wish to try his hand
at the following exercises,
in this connection:*

(1) Show that

- (a) the DUAL of a MODULAR lattice IS MODULAR (use L5, p. 148);
- (b) the DUAL of a DISTRIBUTIVE lattice IS DISTRIBUTIVE (since both L6' and L6" hold in both duals);
- (c) the DUAL of a COMPLEMENTED lattice IS COMPLEMENTED;
- (d) the DUAL of a RELATIVELY COMPLEMENTED lattice IS RELATIVELY COMPLEMENTED;

etc.

(2) Show that there are
5 non-isomorphic posets of
three elements,
three of which are
SELF-DUAL (see pp. 91 and 92).

etc., etc.

This is of course not the place
to go into further details about
the Duality Principle,
but I would like to stress the fact that
this Principle is VERY IMPORTANT
(it cuts the labor of proof
in half! —— see pp. 246ff)
and is used a great deal,
and extends to
other branches of mathematics,
like algebra, projective geometry, logic.

---

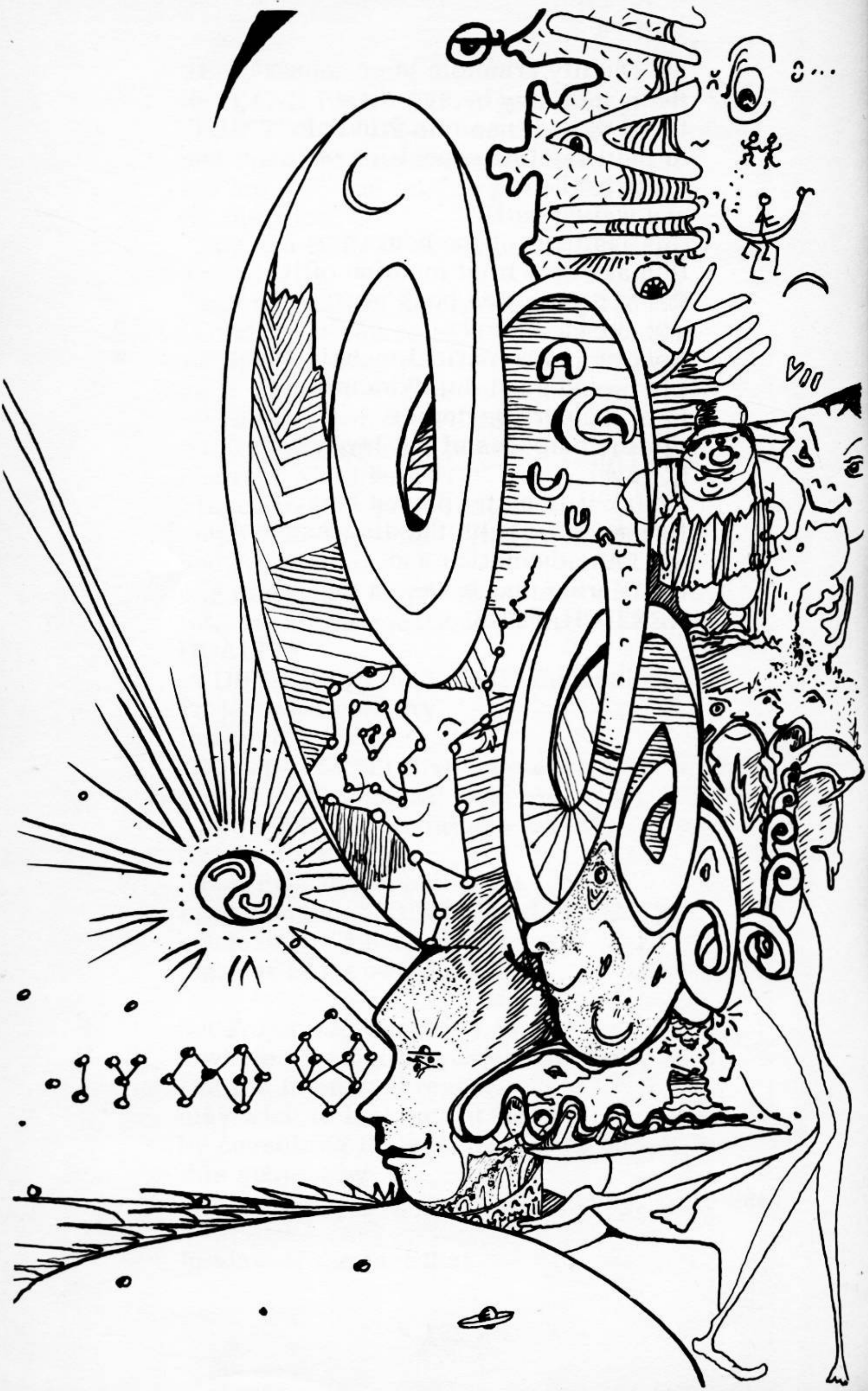
* See also pp. 232 and 233.

Thus, for example,
in PLANE PROJECTIVE geometry,
"POINT" and "LINE" are duals of each other,
and therefore interchangeable
(as are ⌢ and ⌣ on page 246):
for instance,
"Any two POINTS determine a LINE",
and, dually,
"Any two LINES determine a POINT"
(which is of course NOT true in
AFFINE geometry,
as in ordinary Euclidean geometry,
since there two PARALLEL lines
do NOT determine a point, —
contrast APG2 with PG2' on page 192).
Therefore it is important here to
emphasize the word "PROJECTIVE";
and, furthermore,
one must also emphasize the word "PLANE",
for, the above-mentioned statement about
DUALITY
is DIFFERENT in THREE-dimensional
Projective Geometry,
in which
"POINT" and "PLANE" are each other's duals,
and "LINE" is SELF-dual — thus:
"Two POINTS determine a LINE"
has for its dual here:
"Two PLANES determine a LINE",
interchanging "POINTS" and "PLANES"
but leaving "LINE" unchanged,
since it is its own dual.

But Projective Geometry is
a whole fascinating subject in itself,
and the interested reader
may wish to look into it further
by consulting the excellent book on
this subject by
Veblen and Young (see Bibliography, page 286).

In view of the fact that

the Duality Principle is so important,
the reader may be surprised
that I did not use this Principle
to facilitate the exposition of
Lattice Theory,
and waited until
this last part of the book
to make only brief mention of it.
But of course this book is
not intended for
working mathematical technicians
who would need this Principle
from the very beginning.
For the purposes of the lay reader,
however,
I think it is better placed here,
in connection with the discussion of
METAmathematics and
the bearing that it has on the
NEXT CHAPTER.


VII

Read

XXVIII. Life, Liberty, and the Pursuit of Happiness.

Perhaps the reader is startled by
this chapter heading,
and is thinking that
now we are surely
getting away from mathematics
and are going to fall into
the usual traps of
conclusion-jumping and
throw all caution to the winds!

Well,
let us TRY to proceed in
the spirit of mathematics!

First of all
may I recall the
BASIC IMPORTANCE of
that principle of METAmathematics
which DEMANDS that
a set of postulates for ANY system
be CONSISTENT,
that the postulates
DO NOT CONTRADICT each other (see p. 237),
which means of course that
LYING
CANNOT SERVE as an
instrument of thought! *
Now is not this statement
usually considered to be
a MORAL principle?

---

* Since in LYING,
one contradicts something which
is known to be true,
and therefore
if CONTRADICTION is ruled out as
an instrument of thought,
so is LYING also.

And yet
without it we cannot have
ANY satisfactory mathematical system,
nor ANY satisfactory system of thought —
indeed we cannot even
PLAY A GAME properly
with CONTRADICTORY rules!

In a similar way,
I wish to make the point that
there are other important moral ideas
BEHIND THE SCENES,
without which there cannot be
ANY MATHEMATICS or SCIENCE.
And therefore, in this sense,
Science is NOT AT ALL AMORAL —
any more than one could have
a fruitful and non-trivial postulate set in mathematics
which is not subject to
the METAmathematical demand for
CONSISTENCY!

Now, what are some of these other ideas
Behind the Scenes?

Let me suggest only a very few —
and if we accept these,
it will be interesting to see
what follows from them.
Let us take as a basic postulate here:
LIFE.
For without LIFE
there can be
no living thing —
no flowers,
no animals,
no human race —
and also of course
no music, no art,
no science,
NO MATHEMATICS.

I am not suggesting that we consider here
WHETHER life is worth living,
whether it would make more sense
to commit suicide,
whether it is all just
"Sound and Fury, signifying nothing".
I am proposing that
LIFE
be taken as a POSTULATE,
and therefore not subject to proof,
just like any other postulate. *
But I propose to MODIFY this
and take more specifically as
POSTULATE I.:
the preservation of
LIFE for the HUMAN RACE
is a goal of human effort.
This does not mean that
we are to go about
wantonly killing animals
and other living things,
but to do this only when
it is necessary to support
HUMAN LIFE —
for food,
for prevention of disease,
vivisection, etc.
Indeed, a horse or dog or other animals,
through their friendliness and sincerity,
might actually HELP to sustain
Man's spirit and faith and even his life.
And I interpret this postulate
also to mean that
so-called "sports",
like bull-fighting,
or "ganging up" on one little fox —
a whole gang of men and women
(and corrupting even horses and dogs
to help) —
is really a cowardly act,

---

* Please remember that
a POSTULATE is
NOT a "SELF-EVIDENT TRUTH",
but only an ASSUMPTION (see p. 2),
or an UNPROVED THEOREM.

so unsportsmanlike that it is amazing
how this activity could ever be called
a "sport".
All this is by way of interpreting
the meaning to be given to
Postulate I.:
ACCEPTANCE of LIFE for the HUMAN RACE.
Surely everyone will accept the idea that
Life for the Human Race
is definitely present,
behind the scenes,
in science or mathematics.
But this is not all.
For, I take this postulate to mean also
that we are not to limit it to
only a PART of the human race,
as Hitler did,
because this inevitably leads to WAR,
and, in this day of atomic weapons,
this would certainly contradict
Postulate I.,
would it not?
And, if we are trying to suggest
a "SYSTEM" of morality,
as a METAmathematical background,
it must of course itself be CONSISTENT!
May I say at the very outset that
the "SYSTEM" suggested here
makes no pretense of finality (!),
remembering how difficult it is,
EVEN in MATHEMATICS (see pp. 270ff),
to have a postulate set which guarantees
consistency and completeness!
Nevertheless, one must go on (see p. 273),
one must TRY,
one must do one's best, as in science.
And so, let us continue, in all humility,
to try to make
what can only at best be regarded as
tentative suggestions,
in the hope that, the basic idea —
that there is a morality behind the scenes
in Science and Mathematics —

may prove to be helpful
and may be further
improved and strengthened,
as time goes on.
Now,
let us consider a second postulate.
But first may I recall
what physicists call
the two laws of thermodynamics:

1. The Conservation of Energy, namely,
   the total energy in a closed system
   is a CONSTANT —
   it can neither be created nor destroyed,
   but only transformed from
   one kind of energy to another kind —
   thus we can transform
   electrical energy into heat,
   or heat to mechanical energy, etc.,
   without either
   adding to or subtracting from
   the total energy.
   This FIRST LAW of THERMODYNAMICS
   HAS SURVIVED EVEN
   the great and fundamental changes
   made in Physics by
   the Einstein Theory of Relativity!
   For, when we say that
   atomic energy is CREATED,
   we mean that
   the lost mass has been transformed
   into energy,
   since Einstein showed that
   mass and energy are really the same,
   as expressed in his famous equation

   $$E = mc^2$$

   (where E = energy, m = mass, and
   c = the velocity of light).
   Thus even in the case of atomic energy,
   there is no change in
   the TOTAL amount of energy,
   but merely a transformation
   from one kind to another.

2. The SECOND LAW of THERMODYNAMICS
may be stated:
Whenever there is
a transformation or flow of energy,
it happens always
in a certain "natural" DIRECTION,
from a point of higher potential
to one of lower potential,
and never in reverse —
thus water always flows DOWN hill,
heat always flows from
a point of higher temperature
to one of lower temperature,
etc., etc.
And the only way
this direction can be reversed
is by FORCING it
in a direction opposite to
the "natural" flow —
thus water must be FORCED uphill, etc. —
this force being supplied by
a living will.
More generally,
the tendency formulated by the
Second Law —
if not counteracted by other tendencies —
would make the physical world
"run down", "degenerate" to
what is sometimes called "heat death".
Now it seems that
living organisms,
and, in particular, human organisms,
are capable of
introducing directions of change
that oppose
this degeneration, this death,
this DESTRUCTIVE "DEVIL".

May I therefore suggest as

POSTULATE II.:
Each INDIVIDUAL must fight this
"degeneration",
must cling to life as long as possible,

must grow and create —
physically and/or mentally.
And for this we need

POSTULATE III.:

We must all have the LIBERTY to
so grow and create,
without of course interfering
with each other's growth,
which suggests

POSTULATE IV.:

This Freedom must be accompanied by
RESPONSIBILITY,
if it is not to lead to
CONFLICT between individuals which
would contradict the other postulates.

All this is of course very difficult to do,
involving
accepting life without whimpering,
growing without interfering with
the growth of others,
in short
it involves what Goethe called
"cheerful resignation"
("heitere Resignation").
But how can this be done?
It seems clear that we must now add

POSTULATE V.:

The PURSUIT of HAPPINESS *
is a goal of human effort.
For, without some happiness,
or at least the hope of some happiness
(the "pursuit" of happiness)
it would be impossible
to accept "cheerfully"
the program outlined above.
And such acceptance leads to

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* The reader may be interested to see
the discussion on "happiness" in
"Mits, Wits and Logic" (see Bibliography).

a calm, sane performance of our work,
in the spirit in which a mathematician
accepts the postulates of a system
and pursues his creative work
based on these —
accepting even the Great Difficulties
which he encounters
and is determined to conquer.

I repeat that the
"postulates for morality"
given above
are very tentative and inadequate.
What I am trying to emphasize is:

(1) that there IS a morality
behind the scenes
in mathematics and science,
that they are NOT AMORAL;

(2) that this tacit morality
must be made EXPLICIT
in order to be effective —
just as the idea of ORDER
became so effective
when it was made EXPLICIT, * —
and the idea of METAmathematics
is becoming tremendously effective
due to its EXPLICIT formulation; **
that, similarly,
the IMPLICIT postulates of morality
behind mathematics and science
(as suggested by the above discussion)
be made EXPLICIT and EFFECTIVE
and thus free these
wonderful human activities from
the false accusation of being
AMORAL.

And I firmly believe that
the results of such a formulation
will re-discover the conclusions
reached by the

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* See pp. 70ff.

** See pp. 242ff.

great religious leaders and the
great humanitarians.
But, coming from the
mathematical and scientific camp
it will take a form which will be
more accessible to more people,
who may not have
the great INTUITION needed to
arrive at these conclusions
without such scientific formulation.

May I quote here, in support of the above,
from Einstein's "Out of My Later Years":*
in speaking of our religious tradition
which has given us
"the highest principles for
our aspirations and judgments",
he then adds:
"If one were to take that goal out of its
religious form, and look merely at its
purely human side, one might state it
perhaps thus:
free and responsible development of the
individual, so that he may place his
powers freely and gladly in the service
of all mankind."

Do I hear some reader saying
"But how can you say that
Science is NOT AMORAL when
it gives us atomic energy and
DOES NOT CARE whether
we use it for peaceful purposes or
for making bombs!"
To which I say that
since the bomb destroys LIFE,
and without LIFE
there can be NO SCIENCE,
therefore the ideal pursued by
SCIENCE is INCOMPATIBLE with
the consequences that flow from

---

* Published by the
Philosophical Library in 1950,
page 23.

the use of the BOMB.
So it is NOT SCIENCE in Man
but the "devil" in him —
whose objective is DESTRUCTION —
that is consistent with the bomb.
Indeed, does not the "Devil" in Man
use even
RELIGION itself for DESTRUCTION?
And yet
you would not say that
RELIGION is AMORAL,
would you?!

And
to those psychologists who tell us
that there is a deep desire
for CONFLICT
in all of us,
may I say that
the above program is
in complete agreement,
BUT
it is the "degeneration",
the "Devil",
which LIFE MUST FIGHT (p. 259)
and it is NOT our "grandmothers"
whom we wish to murder,
as some people wrongly interpret
this BASIC urge to FIGHT the "Devil".

Thus MAN,
being, on the one hand,
part of the physical world,
is governed by
the Second Law (p. 259),
like any other "thing",
(the "Devil" in him)
BUT, on the other hand,
the LIFE in MAN
(with which METAscience goes along —
Postulate I.)
MUST,
according to Postulate II. (p. 259),

FIGHT this destructive force
within himself.
And this "WAR"
demands our greatest efforts,
so we had better
stop wasting these efforts on
fighting each other
in contradiction to
Postulates III. and IV.
Thus,
the above postulate set,
inadequate as it probably is in
its present form,
is nevertheless, I submit,
behind the scenes in SCIENCE —
in "METAscience" —
and is in agreement with
the basic principles given us by
the great religious leaders
and humanitarians,
and is definitely
NOT a POLLYANNISH program!
It is a TOUGH FIGHT,
which each one of us,
as individuals,
is sure to lose! —
for the Second Law will
surely get us in the end,
by Death —
but which
LIFE itself wins!

If we accept this
"gladly",
as Einstein says (p. 262),
we have that "happiness" (Postulate V.),
without which we could not
"fight the good fight" —
the most interesting and
exhilarating
of all "games"!

How can anyone say that
Science,
with its "METAscience"
behind the scenes,
is AMORAL?!
Is it not rather
the GADGETS themselves,
the BOMBS themselves,
that are AMORAL
(being only "things"),
since they will explode,
indifferently,
to kill millions of people or
to build a new harbor, —
but they are not to be identified with
the spirit in man which is
in his scientific urge!

APPENDIX.

I. Alternative Postulate Sets for a Group.

1. The following set of four postulates are given by Weber: *

   (a) Closure;
   (b) The Associative Law;

   but, instead of (c) and (d) on page 49, Weber takes

   (c)' $\begin{cases} \text{If } ab = ab' \text{ then } b = b' \\ \text{If } ab = a'b \text{ then } a = a' \end{cases}$

   and

   (d)' Given any 2 of the 3 elements, a, b, c, in

   $$ab = c,$$

   the third one can be found.

   From these two, (c)' and (d)',
   he then PROVES that
   the system has
   an Identity element and
   an Inverse for every element.

2. It is interesting to note here that
   a set of ONLY THREE postulates
   for a Group
   was given by Huntington, **
   namely:

   (1) Given any 2 elements a and b, there is an element x such that

   $$a \circ x = b;$$

---

* See the famous Weber's Algebra, vol. II., 1899, pp. 3-4.

** E. V. Huntington: "Simplified Definition of a Group," Bull. of the A.M.S., 8 (1902), pp. 296-300.

(2) Given any 2 elements a and b,
there is an element y such that

$$y \circ a = b;$$

(3) If a, b, c, $a \circ b$, $b \circ c$, and
EITHER $(a \circ b) \circ c$ OR $a \circ (b \circ c)$
are elements of the set, then

$$(a \circ b) \circ c = a \circ (b \circ c).$$

From these very innocent-looking demands,
Huntington then PROVES five theorems,
which he numbers 6 through 10,
and then shows,
in the most SIMPLE and ELEGANT manner
that
his Theorem 10 is the same as
Weber's FIRST postulate (closure),
that
his own Postulate (3) is equivalent to
Weber's SECOND postulate (the Associative Law),
that
his Theorems 6 and 7 contain
Weber's THIRD postulate,
and finally that
his own Postulates 1 and 2,
combined with Theorems 8 and 9,
are equivalent to
Weber's FOURTH postulate.

Thus the two postulate sets are
"strictly equivalent,"
as Huntington says;
and you will surely agree that
Huntington's group postulates are
especially beautiful —
for, as you can see,
he makes so much out of so little,
as does a great artist!

I cannot refrain from adding here that
if you are one of those people who

have sometimes glanced at
the Bulletin of the
American Mathematical Society
and felt that you were
completely beyond your depth,
may I say that
it is not always as formidable
as it looks —
indeed
the article by Huntington referred to above
is not only readable,
even for a novice,
but is really charming in its
simplicity and clarity!
And there are many such.

II. For p. 157:
PROOF of the Theorem:
in a DISTRIBUTIVE LATTICE,
each element has
AT MOST ONE complement:

Thus suppose that
x has TWO complements, u and v;
then
$x \frown u = 0$ and $x \smile u = 1$
and also
$x \frown v = 0$ and $x \smile v = 1$
Therefore
$x \frown u = x \frown v$ and
$x \smile u = x \smile v$.
Consequently
$u \frown (x \smile v) = u \frown (x \smile u)$
$= u \frown (u \smile x)$, by L2, p. 126,
$= u$, by L4.

Furthermore,
since the given lattice is
Distributive,
we have
$u \frown (x \smile v) = (u \frown x) \smile (u \frown v)$
$= (x \frown u) \smile (u \frown v)$ by L2,
$= (x \frown v) \smile (u \frown v)$
(since $x \frown u = x \frown v$,
as shown above.)

But $(v \frown x) \smile (u \frown v) = v \frown (x \smile u)$
because the lattice is Distributive,
and $v \frown (x \smile u) = v$, by L4 again
(remembering that $x \smile u = x \smile v$).

Hence $u = v$ since they have both
been shown above
to be equal to $u \frown (x \smile v)$.
And so the two complements
have been proved to be one and the same!
Hence, in a Distributive Lattice,
an element CANNOT have MORE than ONE
complement.
But of course
it may not have any complement at all,
as in a CHAIN of
more than two elements,
which is
DISTRIBUTIVE but NOT COMPLEMENTED (see p. 162).
Hence an element of a Distributive Lattice has
AT MOST ONE complement. Q.E.D.

III. Some appreciation of
the SUBTLETY of the demands upon
a Postulate Set,
may be obtained from the following:

(1) Alfred Tarski: "Introduction to Logic,"
pp. 136, 137, where he says:
"There are only a few deductive theories known
of which it has been possible to show
that they are
consistent and complete."
And further:
"Perhaps the most interesting example of
a consistent and complete theory
is that supplied by
elementary geometry."
And again:
"The situation changes essentially
as soon as one goes over to such sciences as
arithmetic and advanced geometry.
Probably no one

working in these sciences
doubts their consistency;
and yet as has resulted from
the latest methodological investigations,
a strict PROOF of their CONSISTENCY
meets with great difficulties
of a fundamental nature.
The situation in regard to
the problem of COMPLETENESS
is even worse;
it turns out that
arithmetic and advanced geometry
are INCOMPLETE;
for it has been possible
to set up problems of a
purely arithmetical or geometrical character
that can be
neither positively nor negatively decided
within these disciplines."
And worse yet:
"It might be supposed that this fact
is merely an outcome of
the imperfection of the axiom systems
and methods of proof
at our disposal up to date,
and that a suitable modification
(for instance,
an extension of the axiom system)
may, in the future, yield complete systems.
Deeper investigations, however,
have shown this conjecture to be erroneous:
NEVER will it be possible to build up
a consistent and complete deductive theory
containing as its theorems
all true sentences of
arithmetic or of advanced geometry."
"These exceedingly important achievements
are due to the contemporary Austrian logician,
K. Gödel."

Is this not reminiscent of
Einstein's famous remark:

"Alles was wir machen ist falsch"
(Everything we make is false).
And yet,
let us not forget that,
even though
Mathematics and Science are
NOT PERFECT,
still they have been
eminently successful,
including of course
Einstein's own work.
Thus we have much to learn from
the methods used in these domains,
even while admitting that
NOTHING that MAN does is PERFECT!

To pursue the SUBTLETY further, see

(2) Hilbert and Ackermann:
"Mathematical Logic", pp. 42 and 95,
in which the term
"COMPLETE"
is shown to have
TWO different meanings
(one "stronger" than the other):
"First, it may be taken to mean that
all the true formulas of
a certain domain
which is characterized by content
can be proved from the set of axioms.
However,
the concept of completeness
may also be more STRICTLY formulated,
so that an axiom system
is termed complete ONLY IF
a contradiction always arises when
there is added to the axioms
a formula not previously provable
from them."
So that a system may be
complete in the former sense,
but not in the "stronger" sense of
this term.

Thus, on p. 42, is shown a system,
known as the
"sentential calculus",
which is COMPLETE in BOTH meanings,
whereas, on p. 95,
there is shown a system,
the "restricted predicate calculus"
which is "NOT complete in the
'stronger' sense of the word",
but IS complete in the other sense.
"The proof is due to K. Gödel:
Die Vollständigkeit der Axiome
des logischen Funktionenkalküls.
Mh. Math. Physik vol. 37 (1930)."

It would seem that
these formidable subtleties
might frighten Man away
entirely
from the pursuit of
science and mathematics!
But the great research men
are made of "sterner stuff",
and,
whereas they continue to dig
into these subtle foundations,
they, at the same time,
continue to build the
superstructure —
and with tremendous success!

IV. An Amusing Point.

The reader may be wondering how
(1) the remark on p. 55 that
the cancellation law*
FOLLOWS from the other postulates
in a FIELD,
can be reconciled with

---

* (k) on p. 53:
If $ca = cb$ (with $c \neq 0$)
then $a = b$.

(2) the statement, p. 240,
that Huntington
proved the INDEPENDENCE of
all the 27 postulates given by him for
the algebra of complex numbers,
including that of
the cancellation law! —
which is one of the 27.

First let us see how
(1) is proved:
Given ca = cb (with $c \neq 0$),
show that a = b
WITHOUT using the
cancellation law:
multiplying both sides by c',
which is the
"multiplicative inverse" of c (p. 53),
we obtain

$$c'ca = c'cb$$

or $$(c'c)a = (c'c)b,$$

from which,
since c'c = 1 (p. 53),
and 1 x a = a as in (h) on p. 53
we get a = b.
Now what postulates
did we just use here? —

(a) the existence of
a multiplicative identity, 1,
as in postulate (h), p. 53
and
(b) the existence of
a multiplicative inverse c',
such that c'c = 1
introduced into
the Integral Domain
which made it a FIELD (p. 54).

But,
the algebra of complex numbers
is also a FIELD

and therefore
why does not the cancellation law
FOLLOW from
its other postulates,
as shown above;
and, if so,
how can it be
INDEPENDENT of them?!

The answer to this question
will be most intriguing to
lawyers, chess-players, wrestlers,
and all others who
are interested in
clever ways of
"getting out of a clinch"!
To make this CLEAR
let us take a closer look at
Huntington's postulates:

Postulate 12. There is an element u
in the system,
different from zero,
such that $u \odot u = u$.
Then,
before giving Postulate 13,
he introduces the
Definition. IF there is ONLY ONE
such element u,
this UNIQUE element is called
the UNIT element of the system.
And then
Postulate 13. For every element, a,
in the system
(provided $a \neq 0$),
there is an element a' in the system
such that $a \cdot a' = u$,
where u is the UNIT element.

Now,
in proving the
INDEPENDENCE of

the cancellation law,
Huntington makes up a system*
having 27 postulates,
in which
the cancellation law
does NOT hold,
but the other 26 postulates of
ordinary algebra
ARE SATISFIED —
thus showing that
the cancellation law
is INDEPENDENT of the others!

BUT
note that in this NEW system,
there is MORE THAN ONE u
satisfying Postulate 12,
and therefore,
since u is NOT UNIQUE,
hence, by the Definition above
there is NO UNIT element here,
and consequently,
Postulate 13 DEMANDS NOTHING
(since its demands depend upon
the existence of
a UNIT element)!

Hence,
instead of saying that
Postulate 13 is not satisfied,
mathematicians say instead that
it IS SATISFIED "vacuously".
And this makes good sense,
does it not? —
for,
suppose there were a law that
all women had to wear
"spike" shoes,
then
if you are a man
and prefer to wear

---

* See Trans. A.M.S., vol. VI., 1905, pp. 227-229.

flat heels,
you are NOT breaking the law,
are you? —
but rather you are
"satisfying" the law "vacuously",
since this law
DEMANDS NOTHING of you!

Thus,
since in Huntington's new system
(a) on p. 274 is INOPERATIVE
(since u is NOT UNIQUE),
and, as a result,
(b) is also INOPERATIVE
(since multiplicative inverses
do NOT exist BECAUSE
there is NO UNIT element here),
hence
the cancellation law
DOES NOT FOLLOW here
as it did on p. 274,
and yet,
all the other postulates,
EXCEPT only the cancellation law,
ARE SATISFIED here,
even if, in some cases,
only "vacuously",
as shown above.
This reconciles the two statements
(1) and (2) on pp. 273, 274
which SEEMED contradictory,
but are not,
as you just saw.

However,
Huntington's set of
27 postulates for the algebra of
complex numbers
is of course only
A possible set (see pages 46ff);
indeed it is now considered
preferable to give a set which
does not include his Postulate 13,

and is more in the spirit of
the approach on page 55,
thus avoiding the difficulty described above.

But for those readers who
are interested in the details of
the system invented by Huntington,
mentioned on page 276,
let me describe it as follows:

A. The ELEMENTS α_i $(i = 1, 2, 3, \ldots)$
are COUPLES of REAL numbers:
this idea is
already familiar to you from
an ordinary complex number,

$$a_1 + ib_1$$

in which a_1 and b_1 are
real numbers,
so that
this complex number
may be represented by
the "couple" of real numbers
a_1 and b_1
thus (a_1, b_1).

B. The OPERATIONS,

$$\oplus \quad \text{and} \quad \odot$$

are defined thus:

if $\alpha_1 = (a_1, b_1)$ and $\alpha_2 = (a_2, b_2)$
then

I. their SUM is

$$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

just as for
ordinary complex numbers
for which

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

or, the resulting "couple"

$$(a_1 + a_2, b_1 + b_2)$$

as above.

II. their PRODUCT is

$$(a_1, b_1) \odot (a_2, b_2) =$$

$$(a_1 a_2, b_1 a_2 + a_1 b_2 + b_1 b_2).$$

Note that this PRODUCT
is NOT the same as
the product of
ordinary complex numbers,
since the latter is:

$$(a_1 + ib_1) . (a_2 + ib_2) =$$

$$(a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_1 b_2)$$

or
the "couple"

$$(a_1 a_2 - b_1 b_2, b_1 a_2 + a_1 b_2),$$

which is UNLIKE the couple

$$(a_1 . a_2, b_1 a_2 + a_1 b_2 + b_1 b_2)$$

given above.

Thus the couples used by
Huntington
in this system
are of course
NOT ordinary complex numbers
which DO satisfy the cancellation law,
whereas

we shall soon see that
Huntington's NEW couples
do NOT satisfy the cancellation law,
and yet
the other postulates
DO hold here,
even if only "vacuously" (see p. 277).

And now let us examine

C. Some of the postulates:

since Postulate 12 (p. 275)
mentions "zero",
let us see first
if there is a UNIQUE element z, here
such that

$$z \oplus z = z \text{ (Postulate 5):}$$

let z be the couple (a_1, b_1),
such that $(a_1, b_1) \oplus (a_1, b_1) = (a_1, b_1)$
i.e. $(a_1 + a_1, b_1 + b_1) = (a_1, b_1)$
or $(2a_1, 2b_1) = (a_1, b_1)$,
therefore $2a_1 = a_1$ and $2b_1 = b_1$
or $a_1 = 0$ and $b_1 = 0$.
Hence $z = (0, 0)$,
and, since it is UNIQUE
(there is no other, as you saw above)
therefore z is the zero-element here.

Next, let us see if we have here a u,
different from z, such that

$$u \odot u = u$$

as demanded by Postulate 12, p. 275.
Now let $u = (a_2, b_2)$
such that

$$(a_2, b_2) \odot (a_2, b_2) = (a_2, b_2)$$

then, by the definition of
multiplication on p. 279, we get

$$(a_2^2, \ b_2 a_2 + a_2 b_2 + b_2^2) = (a_2, \ b_2) \qquad (1)$$

therefore

$$a_2^2 = a_2, \ a_2^2 - a_2 = 0, \ a_2(a_2 - 1) = 0$$

so that $a_2 = 0$ or 1.

For $a_2 = 0$, we get, from (1),

$$b_2^2 = b_2$$

therefore $b_2^2 - b_2 = 0$, $b_2(b_2 - 1) = 0$,
so that $b_2 = 0$ or 1;

and for $a_2 = 1$, we get, from (1)

$$b_2 + b_2 + b_2^2 = b_2$$

therefore

$$b_2 + b_2^2 = 0, \ b_2(1 + b_2) = 0$$

which gives $b_2 = 0$ or -1.

Consequently, we have here
the four possibilities

$$\left.\begin{matrix} a_2 = 0 \\ b_2 = 0 \end{matrix}\right\}, \left.\begin{matrix} a_2 = 0 \\ b_2 = 1 \end{matrix}\right\}, \left.\begin{matrix} a_2 = 1 \\ b_2 = 0 \end{matrix}\right\}, \left.\begin{matrix} a_2 = 1 \\ b_2 = -1 \end{matrix}\right\}$$

and since of these
the couple (0, 0) is z,
as shown on p. 280,
we have here then
THREE possible u's, different from z,
namely
$u = (0, 1), \ (1, 0), \ (1, -1)$

all of which satisfy
Postulate 12, p. 275.
For example,
using the definition of multiplication,
given on p. 279,
we have, for $a_2 = 0$ and $b_2 = 1$

$$(0, 1) \odot (0, 1) = (0, 1)$$

since

$$a_2 a_2 = 0, \text{ and } b_2 a_2 + a_2 b_2 + b_2 b_2 = 1,$$

as required.
Similarly show that

$$(1, 0) \odot (1, 0) = (1, 0)$$

and $(1, -1) \odot (1, -1) = (1, -1)$.

And so
there is NOT a UNIQUE u,
hence
multiplicative inverses are
NOT DEMANDED,
and Postulate 13 is
satisfied vacuously,
as was said above.

Thus we could NOT use here
the method on p. 274
to prove that
the cancellation law
follows automatically from the others.
And, indeed,
let us actually show now
that the cancellation law
really does NOT hold here:

i.e. if $\gamma\alpha = \gamma\beta$ $(\gamma \neq 0)$
it does NOT follow here that

$$\alpha = \beta:$$

Let $\gamma = (a_1, b_1)$, $\alpha = (a_2, b_2)$, $\beta = (a_3, b_3)$, then

$\gamma\alpha = (a_1 a_2,\ b_1 a_2 + a_1 b_2 + b_1 b_2)$ and

$\gamma\beta = (a_1 a_3,\ b_1 a_3 + a_1 b_3 + b_1 b_3)$.

Now we have given that

$$\gamma\alpha = \gamma\beta \qquad \text{or}$$

$$(a_1a_2,\ b_1a_2 + a_1b_2 + b_1b_2) =$$

$$(a_1a_3,\ b_1a_3 + a_1b_3 + b_1b_3) \qquad (2)$$

Does it follow that
α must be equal to β?
i.e. that $a_2 = a_3$ and $b_2 = b_3$?

We see from (2) that

$$a_1 a_2 = a_1 a_3$$

and therefore

$$a_2 = a_3 .^*$$

From (2) we also get

$b_1a_2 + a_1b_2 + b_1b_2 = b_1a_3 + a_1b_3 + b_1b_3;$

substituting a_2 for a_3 here, we get

$b_1a_2 + a_1b_2 + b_1b_2 = b_1a_2 + a_1b_3 + b_1b_3$

---

* Note that it is legitimate to cancel the a_1, because the a's are ordinary real numbers, whereas we are about to DISPROVE the cancellation law for the couples like (a_1, b_1) etc. which are elements of Huntington's new system.

or $a_1(b_2 - b_3) = b_1(b_3 - b_2)$ (3)

or $\dfrac{a_1}{b_1} = \dfrac{b_3 - b_2}{b_2 - b_3} = -1$

and therefore

$$a_1 = -b_1 \ .$$

Thus take, for example,

$a_1 = 1$, then $b_1 = -1$, and take also

the two DIFFERENT values

$$b_2 = 2 , \quad b_3 = 3;$$

then, from (3)

$$1(2 - 3) = -1(1)$$

or $$-1 = -1.$$

That is,
(2) can hold WITHOUT $b_2 = b_3$
and therefore
α does NOT have to be equal to β.

In other words,

$$\text{if } \gamma = (1, -1)$$
$$\alpha = (a_2 , 2)$$
$$\beta = (a_2 , 3)$$

then $\gamma\alpha = (a_2 , \ -a_2 + \cancel{2} - \cancel{2})$

and $\gamma\beta = (a_2 , \ -a_2 + \cancel{3} - \cancel{3})$

or $\gamma\alpha = \gamma\beta = (a_2 , \ -a_2)$

and yet

$$\alpha \neq \beta$$

since $\alpha = (a_2 , 2)$
and $\beta = (a_2 , 3)$.

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SPECIAL NOTE:

For those readers who are
particularly interested in the
Foundations of Mathematics
(briefly discussed on pp. 237 ff),
I take pleasure in recommending
two very recent (1958)
and MOST COMPETENT books
(which I received after this book on
LATTICE THEORY
had already gone to press):

(1) RUDOLF CARNAP:
Introduction to Symbolic Logic and its Applications
(Dover Publications, Inc., New York).

(2) A. A. FRAENKEL and Y. BAR-HILLEL:
Foundations of Set Theory
(North-Holland Publishing Co., Amsterdam)